# 3 Mathematical Modeling Equation and Software Selection

The second step in aquifer test modeling is the selection of appropriate aquifer test mathematical modeling equations compatible with the previously defined conceptual model. There are two types of aquifer test mathematical modeling equations:

1. Analytical
2. Numerical

Analytical modeling equations are generally selected when a conceptual model consists of two or less layers and aquifer parameters are fairly uniform in space. Numerical modeling equations are generally selected when a conceptual model consists of more than two layers or the aquifer parameters are highly heterogeneous. It is of interest to note that analytical modeling equation solutions are often verified independently using numerical modeling equations and vice versa.

Both analytical and numerical modeling equations are based on the fundamental principles of conservation of energy, momentum, and mass. These principles and empirical laws are expressed in groundwater flow partial differential equations, which, in turn, are solved analytically or numerically subject to appropriate initial and boundary conditions and source functions to constitute aquifer test mathematical modeling equations. For details concerning principles, empirical laws, partial differential equations, and mathematical modeling equations see Hantush (1964), Remson et al. (1971), Bear (1972), Kinzelbach (1986), Bear and Veruijt (1987), Tien-Chang Lee (1999), and Cheng (2000).

## ANALYTICAL MATHEMATICAL MODELING EQUATIONS

There are two types of analytical mathematical modeling equations:

1. Stehfest algorithm
2. Integral and empirical

Integral and empirical mathematical modeling equations are special cases of Stehfest algorithm mathematical modeling equations with certain parameter values assumed to be negligible. The simulation of wellbore storage and skin, well partial penetration, observation well delayed response, and delayed drainage at the water table is more difficult with integral and empirical modeling equations than with Stehfest algorithm mathematical modeling equations.

Groundwater flow partial differential equations contain a first order differential in time thereby making it possible for the equations to be solved for specified aquifer and well conditions with an integral transform called the Laplace transform. In turn, Laplace transform solutions are analytically or numerically inverted to obtain analytical aquifer test mathematical modeling equations.

Analytical inversion is based on operational calculus (contour integration) and involves complicated semidefinite integrals that are difficult or impossible to evaluate except for simple aquifer and well conditions and, in some instances, short and long elapsed time ranges. For example, analytical inversion under confined leaky with confining unit storativity or unconfined conditions is restricted to certain short and long elapsed time ranges making it difficult to obtain aquifer test mathematical modeling equations covering the entire aquifer test time.

Aquifer test modeling equations based on analytical inversion solutions (herein called the integral and empirical type) are commonly referred to as the Theis or Cooper-Jacob method (pumping test confined nonleaky aquifer), Cooper et al. method (slug test confined aquifer), Hvorslev method (slug test confined nonleaky), Hantush and Jacob method (pumping test confined leaky aquifer), Neuman method (pumping test unconfined aquifer), Bouwer and Rice method (slug test unconfined), and Springer and Gelhar method (slug test high conductivity). Methods are described in detail by Kruseman and de Ridder (1994) and Butler (1998b).

Method assumptions often include fully penetrating wells with no wellbore storage and no aquifer boundaries. In addition, the Hantush and Jacob method assumes the confining unit storativity is negligible. Several elaborate procedures have been developed to determine the best method and graph type (semilogarithmic or double logarithmic) to use with aquifer test data (American Society for Testing Materials [ASTM] guidelines). These guidelines can be downloaded at www.astm.org/cgi-bin/SoftCart.exe/index.shtml?E+mystore.

Method restrictions, simplifying assumptions, and subjective decisions can be largely avoided by using aquifer test mathematical modeling equations based on numerical inversion solutions (herein called the Stehfest algorithm). Many numerical inversion algorithms are available (Davies and Martin, 1979). Algorithms applied to the analysis of aquifer test data (Novakowski, 1990, pp. 99–107) are those developed by Stehfest (1970a, 1970b), Crump (1976), and Talbot (1979). The Stehfest algorithm is the most commonly used algorithm. The Stehfest algorithm is a polynomial approximation and can be used with available groundwater flow Laplace transform equations (Moench and Ogata, 1984, pp. 150–151).

## INTEGRAL AND EMPIRICAL MATHEMATICAL MODELING EQUATIONS

The most commonly used integral aquifer test mathematical modeling equations are frequently labeled:

- Confined nonleaky pumping test — Theis (1935)
- Confined nonleaky slug test — Cooper et al. (1967)
- Confined nonleaky slug test — Hvorslev (1951)
- Confined leaky without confining unit storativity pumping test — Hantush and Jacob (1955)
- Unconfined pumping test — Neuman and Witherspoon (1972) and Neuman (1975a)
- Unconfined slug test — Bouwer and Rice (1976)
- Confined or unconfined high conductivity slug test — McElwee et al. (1992); Springer and Gelhar (1991)

Infrequently used but important integral aquifer test mathematical models are frequently labeled:

- Confined leaky pumping test with confining unit observation wells — Neuman and Witherspoon (1972)
- Induced streambed infiltration pumping test — Rorabaugh (1956) and Zlotnik and Huang (1999)

Other integral aquifer test mathematical modeling equations are described by Kruseman and de Ridder (1994) and Batu (1998).

The integral confined nonleaky pumping test mathematical modeling equation (Theis, 1935, pp. 519–524) is:

$$s = QW(u)/4\pi T \tag{3.1}$$

where

$$u = r^2S/(4Tt) \tag{3.2}$$

$s$ is drawdown, $Q$ is the pumped well discharge rate, $T$ is the aquifer transmissivity, $r$ is the distance from the pumped well, $S$ is the aquifer storativity, and $t$ is the elapsed time.

Major integral confined nonleaky pumping test mathematical modeling equation assumptions are:

- The aquifer is confined by overlying and underlying impermeable deposits.
- There are no aquifer boundaries or discontinuities within the cone of depression.

- The aquifer is homogeneous, isotropic, and of uniform thickness within the cone of depression.
- Prior to pumping, the piezometric surface is horizontal and piezometric pressure is constant.
- The pumping rate is constant.
- The wells penetrate and are open to the entire aquifer thickness so that flow is horizontal and not vertical in the aquifer.
- The wells have infinitesimal diameters and no wellbore storage.
- The wells have no wellbore skin.
- The pumped well has no well loss.
- During pumping, groundwater levels remain above the aquifer top.

$W(u)$ is dimensionless drawdown (well function) and $u$ is dimensionless time. When $u \leq 0.01$ then $W(u) = -0.5772 - \ln(u) = \ln(0.562/u)$ (Hantush, 1964, p. 321). If $u < 0.25$ then $W(u) = \ln(0.78/u)$ and if $u > 1$, then $W(u) = \exp(-1.2u - 0.60)$.

Values of $W(u)$ are commonly calculated with the following polynomial approximation (Abramowitz and Stegun, 1964, p. 231):
when $0 < u \leq 1$

$$W(u) = -\ln u + a_0 + a_1 u + a_2 u^2 + a_3 u^3 + a_4 u^4 + a_5 u^5 \tag{3.3}$$

where

$$\begin{aligned} a_0 &= -0.57721566 & a_3 &= 0.05519968 \\ a_1 &= 0.99999193 & a_4 &= -0.00976004 \\ a_2 &= -0.24991055 & a_5 &= 0.00107857 \end{aligned}$$

when $1 < u < \infty$

$$W(u) = [(u^4 + a_1 u^3 + a_2 u^2 + a_3 u + a_4)/(u^4 + b_1 u^3 + b_2 u^2 + b_3 u + b_4)]/[u\exp(u)] \tag{3.4}$$

where

$$\begin{aligned} a_1 &= 8.5733287401 & b_1 &= 9.5733223454 \\ a_2 &= 18.0590169730 & b_2 &= 25.6329561486 \\ a_3 &= 8.6347608925 & b_3 &= 21.0996530827 \\ a_4 &= 0.2677737343 & b_4 &= 3.9584969228 \end{aligned}$$

The integral confined nonleaky with fully penetrating well slug test mathematical modeling equation (Cooper et al., 1967) is:

$$H/H_0 = W(\alpha,\beta) \tag{3.5}$$

where

$$\alpha = r_w^{\,2} S / r_c^{\,2} \tag{3.6}$$

$$\beta = K_h bt/r_c^2 = (1/u)(r_w^2 S/4r_c^2) \tag{3.7}$$

$H/H_0$ is the dimensionless normalized head in the slugged well, $H$ is the deviation of the head in the slugged well from static conditions, $H_0$ is the initial head change (displacement) in the slugged well, $W(\alpha,\beta)$ is the dimensionless normalized head, $\beta$ is dimensionless time, $r_w$ is the slugged well effective radius, $r_c$ is the slugged well casing radius, $S$ is the aquifer storativity, $K_h$ is the aquifer horizontal hydraulic conductivity, $b$ is the aquifer thickness, and $t$ is the elapsed time.

Major integral confined nonleaky with fully penetrating well slug test mathematical modeling equation assumptions are:

- The aquifer is confined by overlying and underlying impervious deposits.
- There are no aquifer boundaries or discontinuities within the area of influence of the slug.
- The aquifer is homogeneous, isotropic, and of uniform thickness within the area of influence of the slug.
- Prior to the slug test, the piezometric surface is horizontal and piezometric pressure is constant.
- The slugged well penetrates and is open to the entire aquifer thickness so that flow is horizontal and not vertical in the aquifer.
- The slugged well has wellbore storage.
- The slugged well has no wellbore skin.
- During the slug test, groundwater levels remain above the aquifer top.

The Laplace transform solution for Equation 3.5 is presented by Novakowski (1989, p. 2379). The aquifer thickness can be replaced by the effective screen length in the case of a partially penetrating well (Butler, 1998b, pp. 82–87). Partially penetrating slugged well Stehfest algorithm equations were developed by Dougherty and Babu (1984, pp. 1116–1122) and Dougherty (1989, pp. 567–568).

Moench and Hsieh (1985, p. 20) present an equation for analysis of slugged well test data accounting for a skin of finite thickness. The equation assumes a confined nonleaky aquifer and a fully penetrating well. Families of type curves generated with that equation for different values of the ratio of the aquifer hydraulic conductivity to the skin hydraulic conductivity have nearly identical shapes except for very low ratios. Therefore, there is a large degree of nonuniqueness in matching test data to a family of type curves. Accurate estimates of aquifer hydraulic conductivity cannot be obtained under most circumstances, and it is not possible to tell whether there is a skin with a different hydraulic conductivity than that of the aquifer. Butler (1998b, pp. 172–176) describes the Ramey et al. method for analyzing fully penetrating slug test data that takes into account well skin.

The empirical confined nonleaky slug test mathematical modeling equation (Hvorslev, 1951) is:

$$\ln[H(t)/H_0] = -2K_h bt/[r_c^2 \ln(1/(2\Psi) + \{1 + [1/(2\Psi)]^2\}^{0.5})] \tag{3.8}$$

where

$$\Psi = (K_v / K_h)^{0.5} / (b_e / r_w) \qquad (3.9)$$

when the screen bottom is above the aquifer base

$$K_h = r_c^2 \ln(1/(2\Psi) + \{1 + [1/(2\Psi)]^2\}^{0.5})/(2b_e T_b) \qquad (3.10)$$

when the screen bottom abuts the aquifer base

$$K_h = r_c^2 \ln(1/(\Psi) + \{1 + [1/(\Psi)]^2\}^{0.5})/(2b_e T_b) \qquad (3.11)$$

where $K_h$ is the aquifer horizontal hydraulic conductivity, $b$ is the aquifer thickness, $t$ is the elapsed time, $r_c$ is the well casing radius, $K_v$ is the aquifer vertical hydraulic conductivity, $b_e$ is the effective screen length, and $T_b$ is the basic time lag, the time at which a normalized head of 0.368 is obtained.

Major empirical confined nonleaky slug test mathematical modeling equation assumptions are:

- The aquifer is confined by overlying and underlying impervious deposits.
- There are no aquifer boundaries or discontinuities within the area of influence of the slug.
- The aquifer is homogeneous, isotropic, and of uniform thickness within the area of influence of the slug.
- Prior to the slug test, the piezometric surface is horizontal and piezometric pressure is constant.
- The slugged well has wellbore storage.
- The slugged well has no wellbore skin.
- Groundwater levels remain above the aquifer top during the slug test.

The integral confined leaky without confining unit storativity pumping test mathematical modeling equation (Hantush and Jacob, 1955, p. 98) is:

$$s = QW(u,b_c)/4\pi T \qquad (3.12)$$

with

$$u = r^2 S/(4Tt) \qquad (3.13)$$

$$b_c = r/(Tb'/K')^{0.5} \qquad (3.14)$$

$s$ is drawdown, $Q$ is the pumped well discharge rate, $T$ is the aquifer transmissivity, $r$ is the distance from the pumped well to the observation well, $S$ is the aquifer storativity, $t$ is elapsed time, $K'$ is the confining unit vertical hydraulic conductivity, and $b'$ is the confining unit thickness.

Major confined leaky without confining unit storativity pumping test mathematical modeling equation assumptions are:

- The aquifer is confined by an overlying permeable confining unit and an underlying impermeable unit.
- The storativity of the confining unit is negligible.
- There is vertical leakage from the confining unit into the aquifer.
- There are no aquifer boundaries or discontinuities within the cone of depression.
- The aquifer is homogeneous, isotropic, and of uniform thickness within the cone of depression.
- Prior to pumping, the piezometric surface is horizontal and piezometric pressure is constant.
- The pumping rate is constant.
- The wells penetrate and are open to the entire aquifer thickness so that flow is horizontal and not vertical in the aquifer.
- The wells have infinitesimal diameters and no wellbore storage.
- The wells have no wellbore skin.
- The pumped well has no well loss.
- During pumping, groundwater levels remain above the aquifer top.

Values of $W(u,b_c)$ are commonly calculated with the following equations (Wilson and Miller, 1978, p. 505):
when $r/b_c > 2$

$$W(u,b_c) = [\pi/(2r/b_c)]^{0.5}\exp(-r/b_c)\mathrm{erfc}[-(r/b_c - 2u)/(2u^{0.5})] \tag{3.15}$$

Values of erfc($x$) are commonly calculated with the following approximations presented by Abramowitz and Stegun (1964):

$$\mathrm{erfc}(x) = 1/[1 + a_1(x) + a_2(x)^2 + \ldots + a_6(x)^6]^{16} \tag{3.16}$$

$$\mathrm{erfc}(-x) = 1 + \mathrm{erf}(x) \tag{3.17}$$

where

$$\mathrm{erf}(x) = 1 - 1/[1 + a_1(x) + a_2(x)^2 + \ldots + a_6(x)^6]^{16} \tag{3.18}$$

$$\mathrm{erf}(-x) = -\mathrm{erf}(x) \tag{3.19}$$

and

$$\begin{array}{ll} a_1 = 0.0705230784 & a_4 = 0.0001520143 \\ a_2 = 0.0422820123 & a_5 = 0.0002765672 \\ a_3 = 0.0092705272 & a_6 = 0.0000430638 \end{array}$$

when $r/b_c < 2$ and $r/b_c > 0$ and $(r/b_c)^2/(4u) > 5$

$$W(u,b_c) = 2K_0(r/b_c) \quad (3.20)$$

when $r/b_c \leq 2$ and $(r/b_c)^2/(4u) \leq 5$

$$W(u,b_c) = E_1(u) + \sum_{m=1}^{\infty}(1/m!) - [(r/b_c)^2/(4u)]^m E_{m+1}(u) \quad (3.21)$$

with

$$E_1(u) = W(u) \quad (3.22)$$

$$E_{m+1}(u) = (1/m)[\exp(u) - uE_m(u)] \quad (3.23)$$

$$m! = m(2\pi/m)^{0.5} m^m \exp(y) \quad (3.24)$$

with

$$y = 1/(12m) - 1/(360m^3) - m \quad (3.25)$$

Values of $K_0(r/b_c)$ are commonly calculated with the following polynomial approximations presented by Abramowitz and Stegun (1964):

when $0 < r/b_c \leq 2$

$$\begin{aligned}K_0(r/b_c) = {} & -\ln[(r/b_c)/2]I_0(r/b_c) - 0.57721566 + 0.42278420[(r/b_c)/2]^2 \\ & + 0.23069756[(r/b_c)/2]^4 + 0.03488590[(r/b_c)/2]^6 \\ & + 0.0026298[(r/b_c)/2]^8 + 0.00010750[(r/b_c)/2]^{10} \\ & + 0.00000740[(r/b_c)/2]^{12} \quad (3.26)\end{aligned}$$

with

$$\begin{aligned}I_0(r/b_c) = {} & 1 + 3.5156229[(r/b_c)/3.75]^2 + 3.0899424[(r/b_c)/3.75]^4 \\ & + 1.2067492[(r/b_c)/3.75]^6 + 0.2659732[(r/b_c)/3.75]^8 \\ & + 0.0360768[(r/b_c)/3.75]^{10} + 0.0045813[(r/b_c)/3.75]^{12} \quad (3.27)\end{aligned}$$

Equation 3.27 is valid when $-3.75 \leq r/b_c \leq 3.75$.

When $2 < r/b_c < \infty$

$$\begin{aligned}K_0(r/b_c) = {} & \{1.25331414 - 0.07832358[2/(r/b_c)] + 0.02189568[2/(r/b_c)]^2 \\ & - 0.01062446[2/(r/b_c)]^3 + 0.00587872[2/(r/b_c)]^4 - 0.00251540[2/(r/b_c)]^5 \\ & + 0.00053208[2/(r/b_c)]^6\}/[(r/b_c)^{0.5}\exp(r/b_c)] \quad (3.28)\end{aligned}$$

The integral unconfined aquifer pumping test mathematical modeling equation (Neuman, 1975a) is:

$$s = QW(u_a, u_b, \beta, \sigma)/4\pi T \tag{3.29}$$

where

$$u_a = r^2S/(4Tt) \tag{3.30}$$

$$u_b = r^2S_y/(4Tt) \tag{3.31}$$

$$\beta = (r^2K_v)/(b^2K_h) \tag{3.32}$$

$$\sigma = S/S_Y \tag{3.33}$$

$$u_b = \sigma u_a \tag{3.34}$$

$s$ is drawdown, $Q$ is the pumped well discharge rate, $T$ is the aquifer transmissivity, $r$ is the distance from the pumped well to the observation well, $S$ is the aquifer storativity, $t$ is elapsed time, $S_Y$ is the aquifer specific yield, $K_v$ is the aquifer vertical hydraulic conductivity, $K_h$ is the aquifer horizontal hydraulic conductivity, and $b$ is the aquifer thickness.

Major integral unconfined aquifer pumping test mathematical modeling equation assumptions are:

- The aquifer is unconfined at the top and is underlain by impermeable deposits.
- Delayed gravity of upper portions of the aquifer occurs during the pumping period.
- There is instantaneous drainage at the water table.
- The portion of the aquifer dewatered during the pumping period is negligible.
- There are no aquifer boundaries or discontinuities within the cone of depression.
- The aquifer is homogeneous, isotropic, and of uniform thickness within the cone of depression.
- Prior to pumping, the piezometric surface is horizontal and piezometric pressure is constant.
- The pumping rate is constant.
- The wells penetrate and are open to the entire aquifer thickness so that flow is horizontal and not vertical in the aquifer.
- The wells have infinitesimal diameters and no wellbore storage.
- The wells have no wellbore skin.
- The pumped well has no well loss.

The empirical unconfined aquifer slug test mathematical modeling equation (Bouwer and Rice, 1976; Zlotnik, 1994) is:

$$\ln[(H(t)/H_0)] = -\{2K_h b_{sc} t/[r_c^2 \ln(R_c/r_w^*)]\} \tag{3.35}$$

where

$$r_w^* = (K_v/K_h)^{0.5} \tag{3.36}$$

$$\ln(R_c/r_w^*) = \{1.1/\ln[(d + b_{sc})/r_w^*] + D\}^{-1} \tag{3.37}$$

When the well terminates above the aquifer base

$$D = (A + B\{\ln[b - (d + b_{sc})]/r_w^*\})/(b/r_w^*) \tag{3.38}$$

When the term $\{\ln[b - (d + b)]/r_w^*\}$ is greater than 6.0 then the term should be 6.0. When the well terminates at the aquifer base (fully penetrating well)

$$D = C/(b_{sc}/r_w^*) \tag{3.39}$$

$K_h$ is the aquifer horizontal hydraulic conductivity, $r_c$ is the well casing radius, $b$ is the aquifer thickness, $t$ is the elapsed time, $K_v$ is aquifer vertical hydraulic conductivity, $d$ is the $z$ position of the end of the screen, $b_{sc}$ is the screen length.

The empirical coefficients $A$, $B$, and $C$ can be calculated with the following expressions (Boak, 1991; Van Rooy, 1988):

$$\begin{aligned} A = {} & 1.4720 + 3.537 \times 10^{-2}(b_{sc}/r_w^*) - 8.148 \times 10^{-5}(b_{sc}/r_w^*)^2 \\ & + 1.028 \times 10^{-7}(b_{sc}/r_w^*)^3 - 6.484 \times 10^{11}(b_{sc}/r_w^*)^4 \\ & + 1.573 \times 10^{-14}(b_{sc}/r_w^*)^5 \end{aligned} \tag{3.40}$$

$$\begin{aligned} B = {} & 0.2372 + 5.151 \times 10^{-3}(b_{sc}/r_w^*) - 2.682 \times 10^{-6}(b_{sc}/r_w^*)^2 \\ & - 3.491 \times 10^{-10}(b_{sc}/r_w^*)^3 + 4.738 \times 10^{-13}(b_{sc}/r_w^*)^4 \end{aligned} \tag{3.41}$$

$$\begin{aligned} C = {} & 0.7290 + 3.993 \times 10^{-2}(b_{sc}/r_w^*) - 5.743 \times 10^{-5}(b_{sc}/r_w^*)^2 \\ & + 3.858 \times 10^{-8}(b_{sc}/r_w^*)^3 - 9.659 \times 10^{-12}(b_{sc}/r_w^*)^4 \end{aligned} \tag{3.42}$$

The logarithm of the normalized head data is plotted vs. the elapsed time; a straight line is fitted to the data over the time-normalized head interval 0.20 to 0.30; the slope of the line is calculated by estimating the time ($T_0$) at which a normalized head of 0.368 is obtained, and the aquifer horizontal hydraulic conductivity ($K_h$) is estimated with the following equation:

$$K_h = r_c^2[\ln(R_c/r_w^*)]/(2b_{sc}T_0) \tag{3.43}$$

where $r_c$ is the well casing radius and $b_{sc}$ is the screen length.

Equation 3.43 is valid for wells screened below the water table. For wells screened across the water table, a straight line is fitted to the second segment of the time-normalized head graph and the following equation is used to estimate the term $r_c$ (Bouwer, 1989):

$$r_c = [r_{nc}^2 + n(r_{wfp}^2 - r_{nc}^2)]^{0.5} \tag{3.44}$$

where $r_{nc}$ is the nominal radius of the well screen, $r_{wfp}$ is the outer radius of the filter pack, and $n$ is the drainable porosity of the filter pack.

Major integral unconfined aquifer slug test mathematical modeling equation assumptions are:

- The aquifer is unconfined at the top and is underlain by an impermeable unit.
- There is instantaneous gravity drainage of a small upper portion of the aquifer.
- The portion of the aquifer dewatered during the slug test is negligible in comparison to the original aquifer thickness.
- Delayed drainage at the water table is negligible.
- The effects of elastic storage mechanisms are negligible.
- There are no aquifer boundaries or discontinuities within the area of influence of the slug.
- The aquifer is homogeneous, isotropic or anisotropic, and of uniform thickness within the area of influence of the slug.
- Prior to slug test, the water table is horizontal and groundwater levels are constant.
- The slugged well has wellbore storage.
- The slugged well has no wellbore skin.

Slug well test response data are oscillatory in nature in aquifers of very high hydraulic conductivity (Bredehoeft et al., 1966; Van der Kamp, 1976). The *equation for analyzing slugged well test data for formations of very high hydraulic conductivity* is based on solutions to the following equation (Butler, 1998b, p. 155):

$$d^2w_d/dt_d^2 + C_d dw_d/dt_d + w_d = 0 \tag{3.45}$$

where

$$w_d = w/H_0 \tag{3.46}$$

$$t_d = (g/L_c)^{0.5}t \tag{3.47}$$

$$L_c = g/[\omega^2 + (C_v/2)^2] \tag{3.48}$$

$$\omega = 2\pi/(t_{n+1} - t_n) \tag{3.49}$$

$$C_v = 21n(W_n/W_{n+1})/(t_{n+1} - t_n) \tag{3.50}$$

$$C_d = C_v/(g/L_c)^{0.5} \tag{3.51}$$

for confined nonleaky aquifer conditions (McElwee et al., 1992):

$$K_h = (g/L_c)^{0.5}[r_c^2 \ln(1/(2\Psi) + \{1 + [1/(2\Psi)]^2\}^{1/2})/(2\ C_d b_{sc})] \tag{3.52}$$

where

$$\Psi = (K_v/K_h)^{0.5}/(b_e/r_w) \tag{3.53}$$

for unconfined aquifer conditions (Springer and Gelhar, 1991):

$$K_h = (g/L_c)^{0.5}[r_c^2 \ln(R_c/r_w^*)/(2C_d b_{sc})] \tag{3.54}$$

$\ln[(R_c/r_w^*)]$ is patterned after a like term in the Bouwer and Rice method, $g$ is the acceleration due to gravity (9.754 m/sec/sec or 32 ft/sec/sec), $w$ is the deviation of water level from static level in the slugged well, $L_c$ is the effective column length, $C_d$ is the dimensionless damping parameter, $C_v$ is the damping parameter, $\omega$ is the frequency parameter, $t_n$ is the time of the $n$th peak or trough in the slugged well data, $b_e$ is the effective screen length, $b_{sc}$ is the screen length, $r_w$ is the well effective radius, $K_v$ is the aquifer vertical hydraulic conductivity, $K_h$ is the aquifer horizontal hydraulic conductivity, $H_0$ is the initial head change (displacement) in the slugged well, and $W_n$ is the $w$ value at the $n$th peak or trough in the slugged well data. The damping and frequency parameters are estimated from subsequent peaks or troughs in the slugged well data.

Major slugged well test data for formations of very high hydraulic conductivity mathematical modeling equation assumptions are:

- The aquifer has a high conductivity and is either confined by overlying and underlying impermeable units or unconfined at the top and underlain by impervious units.
- Slug impacts are similar to the behavior of a damped spring.
- There are no aquifer boundaries or discontinuities within the area of influence of the slug.
- The aquifer is homogeneous, isotropic or anisotropic, and of uniform thickness within the area of influence of the slug.
- Prior to the slug test, the piezometric surface or water table is horizontal and piezometric pressure or the groundwater level is constant.

- The slugged well has wellbore storage.
- The slugged well has no wellbore skin.

Sometimes, a confined leaky aquifer system has a very low confining unit vertical hydraulic conductivity. Under this condition, it is often impossible to determine confining unit parameter values using data for wells in the aquifer because the effects of leakance are too small during a normal pumping test period to be analyzed with any reasonable degree of accuracy. However, it is possible to determine the aquifer parameter values using the aquifer well data.

When the confining unit has a very low vertical hydraulic conductivity, a piezometer is constructed in the confining unit a few feet above the aquifer top and at the same location as one of the aquifer observation wells. The nested aquifer observation well and the confining unit piezometer must be close to the pumped well and water levels in the aquifer observation well and the confining unit piezometer must be measured at the same elapsed time.

The *integral confined leaky pumping test with confining unit observation wells* (Neuman and Witherspoon, 1972) mathematical modeling equation is:

$$s_c = QW(u,u_c)/(4\pi T) \tag{3.55}$$

where

$$u_c = z^2 S'/(4K'b't) \tag{3.56}$$

$s_c$ is the drawdown in the confining unit, $Q$ is the pumped well discharge rate, $T$ is the aquifer transmissivity, $z$ is the vertical distance from the aquifer top to the base of the confining unit piezometer, $S'$ is the confining unit storativity, $K'$ is the confining unit vertical hydraulic conductivity, $b'$ is the confining unit thickness, and $t$ is the elapsed time.

Assuming that an aquifer observation well and a confining unit piezometer are located at the same short radial distance (< 300 ft) from the pumped well and drawdowns in the aquifer observation well and the confining unit piezometer are measured at the same elapsed time, the ratio of the drawdown in the confining unit ($s_c$) and the drawdown ($s$) in the aquifer is:

$$s_c/s = W(u,u_c)/W(u) \tag{3.57}$$

Values and curves of $W(u,u_c)/W(u)$ versus $1/u_c$ for different values of $u$ are given by Kruseman and de Ridder (1994, pp. 94–95). A value of $1/u_c$ is interpreted from the values or curves based on Equation 3.55, the measured $s_c/s$ ratio, and a previously determined value of $u$ for the aquifer. The ratio $S/K$ is calculated with Equation 3.56.

The *integral induced streambed infiltration pumping test* (Rorabaugh, 1956; Zlotnik and Huang, 1999) mathematical modeling equations are commonly used to estimate aquifer parameter values and the streambed vertical hydraulic conductivity. During an induced infiltration pumping test, a well near a stream is pumped and drawdowns are measured in several nearby observation wells on a ray through the pumped well and parallel to the stream. Prior to the test, water level changes in an observation well near the stream due to stream stage changes are measured. This data is used to determine the streambed leakance.

Aquifer transmissivity is calculated with distance-drawdown data for observation wells on a ray through the pumped well and parallel to the stream measured at the end of the test after groundwater levels stabilize. These observation wells are approximately equidistant from the recharging image well simulating the stream. Thus the effects of induced infiltration on water levels in these wells are approximately equal and the hydraulic gradient of the cone of depression near the production well and parallel to the stream is not distorted to any appreciable degree. A plot of drawdown in the observation wells parallel to the stream vs. the logarithm of the distances between the pumped and observation wells yields a straight line. The slope of the straight line and the pumping rate are inserted in the following equation to calculate aquifer transmissivity (Cooper and Jacob, 1946, pp. 526–534):

$$T = 2.3Q/(2\pi\Delta s) \tag{3.58}$$

where $T$ is the aquifer transmissivity, $Q$ is the pumped well discharge rate, and $\Delta s$ is the drawdown per logarithmic cycle (slope of the straight line).

The distance between the pumped well the recharging image well simulating the stream is calculated with the following equation (see Rorabaugh, 1956, pp. 101–169):

$$2a = \exp(c)r \tag{3.59}$$

where

$$c = 2\pi Ts/Q \tag{3.60}$$

$T$ is the aquifer transmissivity, $s$ is drawdown, $Q$ is the pumped well discharge rate, $a$ is the distance between the pumped well and the effective line of recharge, and $r$ is the distance between a particular observation well and the pumped well.

Several values of aquifer specific yield are assumed and observation well drawdowns for each assumed value are calculated with the following equation (Ferris et al., 1962, pp. 144–166):

$$s_o = [Q/(4\pi T)]W_b(u) \tag{3.61}$$

where

$$W_b(u) = W(u) - W(u_i) \tag{3.62}$$

$$u = r^2 S_Y/(4Tt) \quad (3.63)$$

$$u_i = r_i^2 S_Y/(4Tt) \quad (3.64)$$

$s_o$ is the calculated drawdown in an observation well with the assumed aquifer specific yield $S_Y$ and previously calculated aquifer transmissivity $T$, $Q$ is the pumped well discharge rate, $r$ is the distance between the pumped well and an observation well, $r_i$ is the distance between an observation well and the recharge image well simulating the stream, and $t$ is the elapsed time.

Calculated values of drawdown are compared with measured drawdowns and that specific yield used to calculate drawdowns equal to the measured drawdowns is assigned to the aquifer.

The streambed leakance is estimated with the previously calculated values of aquifer transmissivity and specific yield, measured water level changes in an observation well near the stream due to stream stage changes, and the following equations (Zlotnik and Huang, 1999, pp. 599–605):

$$s_{sc} = \mathrm{erfc}[(x' - 1)/(2t'^{0.5})] - \exp[\xi(x' - 1) + t'\xi^2]$$
$$\mathrm{erfc}[(x' - 1)/(2t'^{0.5}) + \xi\, t'^{0.5}] \quad (3.65)$$

where

$$x' = x/w \quad (3.66)$$

$$t' = Tt/(S_Y w^2) \quad (3.67)$$

$$\xi = \gamma^{0.5} \tanh \gamma^{0.5} \quad (3.68)$$

$$\gamma = K_s' w^2/(b_s' T) \quad (3.69)$$

$s_{sc}$ is the water level change in an observation well per unit stream stage change, $T$ is the aquifer transmissivity, $S_Y$ is the aquifer specific yield, $x$ is the distance from the streambed center to an observation well, $w$ is the half-width of the streambed, $t$ is the elapsed time after a stream stage change began, $K_s'$ is the streambed vertical hydraulic conductivity, $b_s'$ is the streambed thickness, and $K_s'/b_s'$ is the streambed leakance. Materials beneath the streambed are assumed to be saturated and any seepage of water through unsaturated materials beneath the streambed is assumed to be negligible. If materials beneath the streambed are unsaturated, calculated streambed leakance will be less than actual leakance because the influence of negative pressure heads in unsaturated materials is ignored. In many cases, inaccuracies in estimating streambed leakance may overshadow errors due to ignoring unsaturated conditions (see Peterson, 1989, pp. 899–927).

Values of $s_{sc}$ for selected values of streambed leakance are calculated and compared with the measured water level changes per unit stream stage change. The streambed leakance that results in an acceptable match of calculated and measured water level changes per unit stream stage change is assigned to the stream.

## Wellbore Storage Effects

Some integral aquifer test modeling equations assume pumped and observation wellbore storage are negligible. If the conceptual model wells have wellbore storage, dimensionless time-drawdown values calculated with the integral aquifer test modeling equations should be adjusted for wellbore storage effects before they are used to calculate drawdown values. Otherwise, the ranges of pumping test data that are not likely to be affected appreciably by wellbore storage are subjectively selected (filtered) for analysis. Wellbore storage adjustments are the differences between dimensionless time drawdown with and without wellbore storage.

In general, pumped wellbore storage adjustments tend to be negligible except for the first few minutes of a pumping test with moderate to high (> 1000 $ft^2$/day) transmissivities and small (< 0.5 ft) well radii as illustrated in Figure 3.1. Appreciable pumped wellbore storage adjustments are required during the first several hours or even days of the pumping test with lower transmissivities or large well radii or small storativities. Adjustments for observation wellbore storage increase as the distance from the production well and storativity decreases and the production or observation well radius increases.

Adjustments for observation wellbore storage are required close to the pumped well (within tens of feet) during early elapsed times under most aquifer

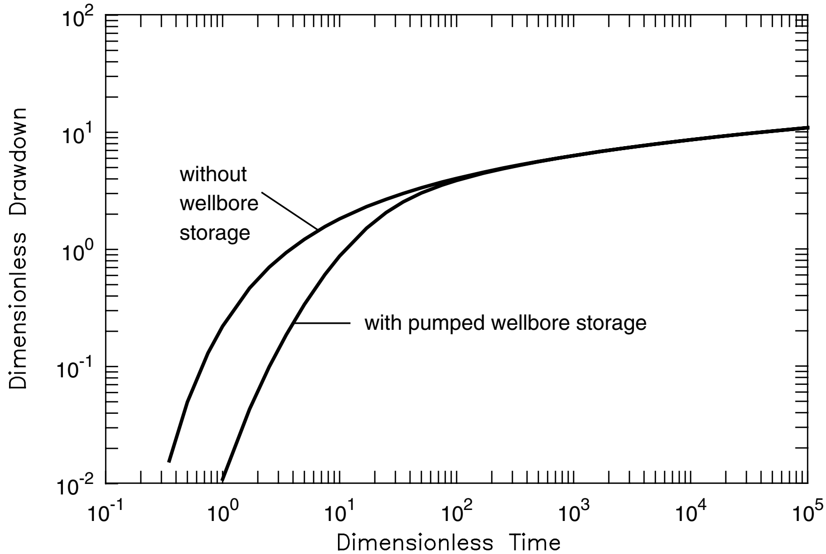



**FIGURE 3.1** Graph showing wellbore storage effects on pumping test type curve values.

conditions. Both pumped and observation wellbore storage adjustments decrease with time (Fenske, 1977). Commonly, with storativity in the $1 \times 10^4$ range and at distances exceeding several tens of feet beyond the production well, the effect of observation wellbore is negligible and adjustments are not required (Fenske, 1977; Tongpenyai and Raghavan, 1981).

The effects of pumped and observation wellbore storage are dependent on the relative quantities of discharge derived from well storage and aquifer storage. During early elapsed times when change in drawdown is rapid, a large portion of the discharge is derived from well storage. During later elapsed times when change in drawdown is very slow, a large portion of the discharge is derived from aquifer storage. Observation wells contain a significant quantity of stored water. With the start of pumping, rapid changes in hydraulic head in the aquifer may not be accurately reflected by measurements in observation wells because of the finite time it takes to dissipate stored water and reach equilibrium with the hydraulic head in the aquifer.

The ratio of the discharge derived from aquifer storage ($Q_{aw}$) and the discharge rate ($Q$) for infinite confined nonleaky aquifers with fully penetrating wells is as follows (Fenske, 1977, p. 90):

without pumped and observation wellbore storage

$$Q_{aw}/Q = 1 \tag{3.70}$$

with pumped wellbore storage

$$\begin{aligned} Q_{ap}/Q = 1 - Q_w/Q = 1 - 1/\alpha\ \{&[e^{-n} + W(n)]/[(ne^{n})^{-1} - (1 - 1/\alpha)W(n)] \\ &- W(n)[e^{-n}(1/n + 1/\alpha)]/[(ne^{-n})^{-1}(1 - 1/\alpha)W(n)]^2\} \end{aligned} \tag{3.71}$$

with pumped and observation wellbore storage

$$\begin{aligned} Q_{apo}/Q = 1 - Q_w/Q = 1 - 1/\alpha\{&[e^{-n} + W(n)]/[(ne^{n})^{-1} - (1 - 1/\alpha)W(n) \\ &+ (1/\beta_m)W(m)] - W(n)[e^{-n}\ (1/n + 1/\alpha) + (1/\beta_m)e^{-m}]/[(ne^{n})^{-1} \\ &- 1(1 - 1/\alpha)W(n) + (1/\beta_m)W(m)]^2\} \end{aligned} \tag{3.72}$$

where

$$n = Sr_w^2/(4Tt) \tag{3.73}$$

$$m = Sr^2/(4Tt) \tag{3.74}$$

$$\alpha = [r_w^2/(r_c^2 - r_d^2)]S \tag{3.75}$$

$$\beta_m = (r_w^2/r_o^2)S/(1 - S) \tag{3.76}$$

$Q_{aw}$ is the discharge from the aquifer without pumped and observation wellbore storage, $Q_{ap}$ is the discharge from the aquifer with pumped wellbore storage, $Q_{apo}$ is the discharge from the aquifer with pumped and observation wellbore storage $Q$ is the total discharge, $S$ is the aquifer storativity, $r_w$ is the pumped well effective radius, $T$ is the aquifer transmissivity, $t$ is the elapsed time, $r$ is the distance from the pumped well to the observation well, $r_c$ is the pumped well casing radius, $r_p$ is the pump pipe radius, and $r_o$ is the observation well casing radius.

Values of dimensionless drawdown $W(n)$ and $W(m)$ are commonly calculated with a polynomial approximation previously described for $W(u)$ (Abramowitz and Stegun, 1964, p. 231).

Both dimensionless time and dimensionless drawdown values are affected by wellbore storage. Dimensionless time values for early elapsed times are shifted to the right (dimensionless time values increase) due to wellbore storage effects and dimensionless drawdown values for early elapsed times are shifted downward (dimensionless drawdown values decrease) due to wellbore storage effects. Dimensionless time adjustments for wellbore storage effects can be calculated with the following equations (Fenske, 1977, p. 89):

The dimensionless time adjustment for pumped wellbore storage $1/u_{adp}$ is:

$$1/u_{adp} = 1/u_{pw} - 1/u \tag{3.77}$$

The dimensionless time adjustment for observation wellbore storage $1/u_{ado}$ is:

$$1/u_{ado} = 1/u_{pow} - 1/u_{pw} \tag{3.78}$$

The dimensionless time adjustment for both pumped and observation wellbore storage $1/u_{adpo}$ is:

$$1/u_{adpo} = 1/u_{pow} - 1/u \tag{3.79}$$

where:

dimensionless time without wellbore storage

$$1/u = 4Tt/(r^2S) \tag{3.80}$$

dimensionless time with pumped wellbore storage

$$1/u_{pw} = (Q_{ap}/Q)[(ne^n)^{-1} - (1 - 1/\alpha)W(n)] \tag{3.81}$$

dimensionless time with both pumped and observation wellbore storage

$$1/u_{pow} = (Q_{apo}/Q)[(ne^n)^{-1} - (1 - 1/\alpha)W(n) + (1/\beta_m)W(m)] \tag{3.82}$$

Dimensionless drawdown adjustments for wellbore storage effects can be calculated with the following equations (Fenske, 1977, p. 89):

The dimensionless drawdown adjustment for pumped wellbore storage $W(u)_{adp}$ is:

$$W(u)_{adp} = W(u)_{pw} - W(u) \tag{3.83}$$

The dimensionless drawdown adjustment for observation wellbore storage $W(u)_{ado}$ is:

$$W(u)_{ado} = W(u)_{pow} - W(u)_{pw} \tag{3.84}$$

The dimensionless drawdown adjustment for both pumped and observation wellbore storage $W(u)_{adpo}$ is:

$$W(u)_{adpo} = W(u)_{pow} - W(u) \tag{3.85}$$

where:

dimensionless drawdown without wellbore storage

$$W(u) \tag{3.86}$$

dimensionless drawdown with pumped wellbore storage

$$W(u)_{adp} = (Q_{ap}/Q)W(u) \tag{3.87}$$

dimensionless drawdown with pumped and observation wellbore storage

$$W(u)_{apo} = (Q_{apo}/Q)W(u) \tag{3.88}$$

Major wellbore storage adjustment mathematical modeling equation assumptions are:

- The aquifer is confined by overlying and underlying impermeable units.
- There are no aquifer boundaries or discontinuities within the cone of depression.
- The aquifer is homogeneous, isotropic, and of uniform thickness within the cone of depression.
- Prior to pumping, the piezometric surface is horizontal and piezometric pressure is constant.
- The pumping rate is constant.
- The wells penetrate and are open to the entire aquifer thickness so that flow is horizontal and not vertical in the aquifer.
- The wells have infinitesimal diameters and no wellbore storage.
- The wells have no wellbore skin.
- The pumped well has no well loss.
- During pumping, groundwater levels remain above the aquifer top.

Although the dimensionless time and dimensionless drawdown adjustments strictly apply to confined nonleaky aquifers, they can be applied to other aquifer conditions with little error because the adjustments are usually of significance only during early elapsed times when confined nonleaky conditions prevail under most aquifer conditions.

## Well Partial Penetration Effects

Some integral aquifer test mathematical modeling equations assume fully penetrating wells. If the conceptual model wells partially penetrate the aquifer, dimensionless time drawdowns should be adjusted for the effects of partially penetrating wells values, otherwise aquifer test data that are not likely to be affected appreciably by partial penetration effects are subjectively selected for analysis. Partially penetrating pumped wells induce vertical components of flow that are assumed to be negligible in some aquifer test modeling equations. Well partial penetration adjustments are strongest at the pumped well face and decrease with increasing distance from the pumped well.

Well partial penetration adjustments may be either negative or positive depending on well geometry as illustrated in Figure 3.2. For example, if the pumped and observation wells are both open in either the top or bottom portion of the aquifer, the measured drawdown in the observation well is greater than it would be with fully penetrating wells. If the pumped well is open to the top of the aquifer and the observation well is open to the bottom of the aquifer, or vice versa, the measured drawdown in the observation well is less than it would be with fully penetrating wells.

The distance beyond which well partial penetration adjustments are negligible is defined by the following equation (Hantush, 1964, p. 351):

$$r_{pp} = 2b(K_h/K_v)^{0.5} \tag{3.89}$$

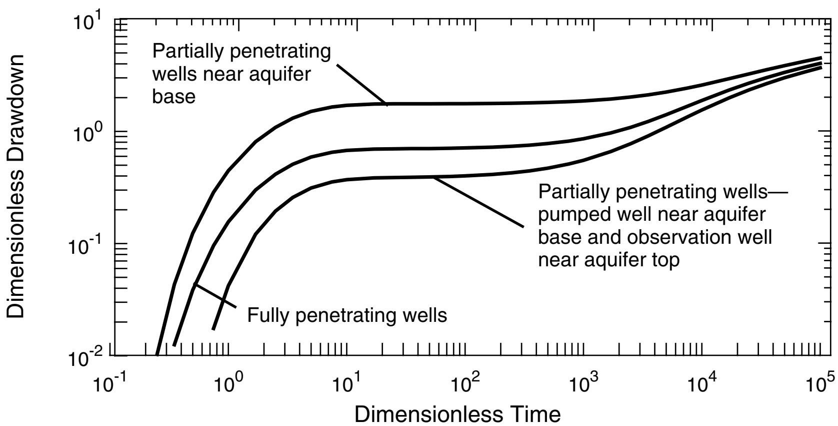



**FIGURE 3.2** Graph showing well partial penetration effects on pumping test type curve values.

$r_{pp}$ is the distance beyond which the effects of well partial penetration are negligible, $b$ is the aquifer thickness, $K_h$ is the aquifer horizontal hydraulic conductivity, and $K_v$ is the aquifer vertical hydraulic conductivity.

Drawdown during the entire elapsed time in confined aquifers or during early and late elapsed times, but not intermediate elapsed times, in unconfined aquifers due to the effects of well partial penetration can be calculated with the following equations (Hantush, 1961, pp. 85 and 90; Reed, 1980, pp. 8–10):

$$s_{pp} = QW_{pp}(u\ \ldots\ )/(4\pi T) \tag{3.90}$$

with

$$W_{pp}(u\ \ldots\ ) = 2b^2/[\pi^2(L - D)(L' - D')] \sum_{n=1}^{\infty} (1/n^2)[\sin(n\pi L/b) - \sin(n\pi D/b)]$$
$$[\sin(n\pi L'/b) - \sin(n\pi D'/b)]W(u,b_{pp}) \tag{3.91}$$

where

$$b_{pp} = (K_v/K_h)^{0.5} n\pi r/b \tag{3.92}$$

$s_{pp}$ is the drawdown due to the effects of well partial penetration, $T$ is the aquifer transmissivity, $Q$ is the pumped well discharge rate, $b$ is the aquifer thickness, $L$ is the vertical distance from the aquifer top to the bottom of the pumped well screen, $D$ is the vertical distance from the aquifer top to the top of the pumped well screen, $L'$ is the vertical distance from the aquifer top to the bottom of the observation well screen, $D'$ is the vertical distance from the aquifer top to the top of the observation well screen.

Well partial penetration adjustments increase during early elapsed times. For larger times $t > b^2S/[2(K_v/K_h)T]$ or $t > bS/(2K_v)$, well partial penetration adjustments gradually level off, become constant in time, and are equal to $2K_0(b_{pp})$.

Dimensionless drawdown values for fully penetrating conditions can be converted to dimensionless drawdown values for partially penetrating wells using the following equation:

$$W_p(u\ldots) = W(u\ldots) + W_{pp}(u\ldots) \tag{3.93}$$

where $W_p(u\ldots)$ is the dimensionless drawdown with well partial penetration effects, $W(u\ldots)$ is the dimensionless drawdown without the effects of well partial penetration, and $W_{pp}(u\ldots)$ is the dimensionless drawdown due to the effects of well partial penetration.

Partial penetration dimensionless drawdown values for the pumped well are calculated by substituting $r_w$ for $r$ and $L' = (L + D)/2$ and $D' = L'\ [0.1(L' - D)]$ (see Hantush, 1964, p. 352).

Major well partial penetration mathematical modeling equation assumptions are:

- The aquifer is confined by overlying and underlying impermeable units.
- There are no aquifer boundaries or discontinuities within the cone of depression.
- The aquifer is homogeneous, isotropic, and of uniform thickness within the cone of depression.
- Prior to pumping, the piezometric surface is horizontal and piezometric pressure is constant.
- The pumping rate is constant.
- The wells penetrate and are open to the entire aquifer thickness so that flow is horizontal and not vertical in the aquifer.
- The wells have infinitesimal diameters and no wellbore storage.
- The wells have no wellbore skin.
- The pumped well has no well loss.
- During pumping, groundwater levels remain above the aquifer top.

## Pumped Well Conditions

Both integral and Stehfest algorithm aquifer test modeling equations assume that nonlinear well losses in the pumped well are negligible. If the conceptual model pumped well has well loss, drawdown in the pumped well should be adjusted for the effects of nonlinear well loss before analyzing data for the pumped well. Nonlinear well losses occur inside the well screen, in the suction pipe, and in the zone adjacent to the well where the flow is turbulent.

Well loss can be estimated with the following equation (Jacob, 1947):

$$s_{wL} = CQ^2 \tag{3.94}$$

where $s_{wL}$ is the component of drawdown in the pumped well due to well loss, $C$ is the well loss constant, and $Q$ is the pumped well discharge rate.

Typical well loss constants are (Walton, 1962, p. 27): negligible well loss — 0 sec$^2$/ft$^5$, low well loss — 1 sec$^2$/ft$^5$, moderate well loss — 5 sec$^2$/ft$^5$, and severe well loss — 20 sec$^2$/ft$^5$. The well loss constant is usually estimated by conducting a step-drawdown test. The well is pumped at three or more constant fractions of full capacity for periods of one hour and drawdowns are measured during each period. Assuming that the well is stable and well loss is equal to $CQ^2$, the well loss constant $C$ can be estimated with the following equation (Jacob, 1947):

$$C = (\Delta s_i/\Delta Q_i - \Delta s_{i-1}/\Delta Q_{i-1})/(\Delta Q_i + \Delta Q_{i-1}) \tag{3.95}$$

where $\Delta s_i$ is the increment of drawdown at the end of pumping period $i$ due to the increment of discharge $\Delta Q_i$ during pumping period $i$.

For Step 1 and Step 2:

$$C = (\Delta s_2/\Delta Q_2 - \Delta s_1/\Delta Q_1)/(\Delta Q_1 + \Delta Q_2) \tag{3.96}$$

For Step 2 and Step 3:

$$C = (\Delta s_3/\Delta Q_3 - \Delta s_2/\Delta Q_2)/(\Delta Q_2 + \Delta Q_3) \tag{3.97}$$

If the well is unstable, $C$ cannot be calculated with data for Step 2 and Step 3 because the calculations yield a negative $C$. In this case, data for Step 1 and Step 2 are combined and $C$ is estimated with the following equation (Jacob, 1947; Walton, 1991, p. 165):

For Step 1 plus Step 2 and Step 3:

$$C_{1+2\text{ and }3} = (\Delta s_3/\Delta Q_3 - \Delta s_{1+2}/\Delta Q_{1+2})/(\Delta Q_{1+2} + \Delta Q_3) \tag{3.98}$$

where $\Delta s_3$ is the increment of drawdown during the third pumping period with an increment of discharge rate $\Delta Q_3$, $\Delta s_{1+2}$ is the increment of drawdown during the first pumping period plus the increment of drawdown during the second pumping period, and $\Delta Q_{1+2}$ is the increment of discharge rate during the first pumping period plus the increment of discharge rate during the second pumping period.

There are more sophisticated well loss equations that may or may not improve the precision of estimates (Rorabaugh, 1953; Lennex, 1966; Hantush, 1964; Bierschenk, 1963; Eden and Hazel, 1973).

## Aquifer Boundary Effects

Integral aquifer test modeling equations assume that the aquifer is infinite in areal extent. If the conceptual model aquifer is finite, drawdown should be adjusted for boundary effects as illustrated in Figure 3.3, otherwise, ranges of data not likely to be affected by boundaries must be subjectively selected for analysis.

The existence of hydrogeologic boundaries (full or partial barrier or recharge) can limit the continuity of an aquifer in one or more directions. Partial barrier or recharge boundaries are called discontinuities. Pumping test data will show the impacts of a full boundary when transmissivity in the immediate vicinity of the pumping well is ten times greater than or one tenth less than the transmissivity at some distance from the pumping well (Fenske, 1984, pp. 131–132). Adjustments for boundary effects are made with the image well theory to be explained later.

## Noordbergum Confined Leaky Aquifer Effect

When water is pumped from a confined leaky aquifer, the head in the overlying or underlying confining unit can rise. This effect (called the Noordbergum effect) is attributed to three-dimensional deformation of the aquifer and confining unit when pumping starts and is not considered in integral and Stehfest algorithm

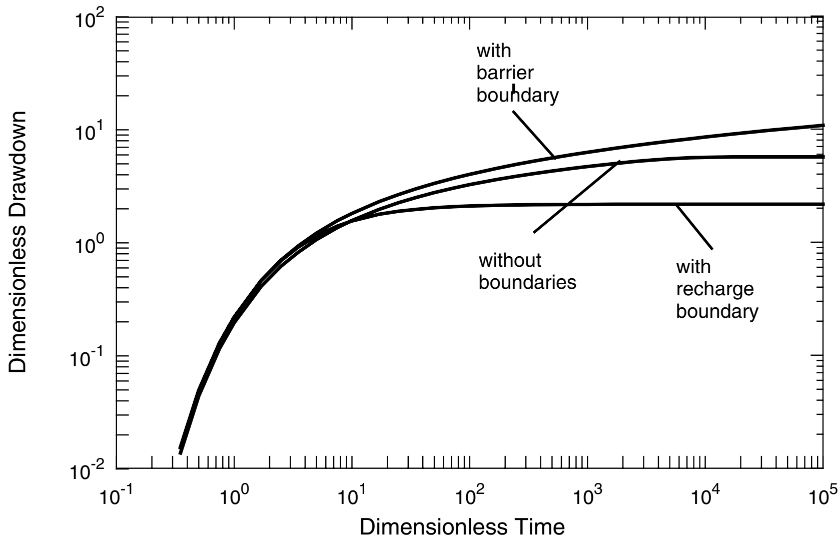



**FIGURE 3.3** Graph showing boundary effects on pumping test type curve values.

aquifer test modeling equations. Analyzing this effect requires the coupling of fluid flow and aquifer deformation (see Verruijt, 1969; Rodriques, 1983; Hsieh, 1996; Kim and Parizak, 1997; and Burbey, 1999).

## STEHFEST ALGORITHM PUMPING TEST MATHEMATICAL MODELING EQUATIONS

Available Stehfest algorithm pumping test mathematical modeling equations cover infinite and finite aquifers, fully or partially penetrating wells with or without wellbore storage and skin, and the following aquifer conditions:

- Confined nonleaky
- Confined leaky without confining unit storativity
- Confined leaky with confining unit storativity and source unit above confining unit
- Confined leaky with confining unit storativity and impermeable unit above confining unit
- Confined leaky with confining unit storativity and water table within confining unit
- Confined fissure and block (double porosity) with slab-shaped blocks
- Confined fissure and block (double porosity) with sphere-shaped blocks
- Unconfined with or without delayed drainage at water table
- Induced streambed infiltration

Stehfest algorithm pumping test mathematical models are based on Laplace transforms $F(p)$ of groundwater functions $f(t)$ (Moench and Ogata, 1984, p. 150; Tien-Chang Lee, 1999, p. 80; Cheng, 2000, pp. 91–93). $f(t)$ can be calculated at any dimensional time $t > 0$ with a number of discrete $F(p)$ values using the following approximation equation:

$$f(t) \approx [(\ln 2)/t] \sum_{i=1}^{N} V_i F[\mathrm{i}(\ln/2)/t)] \tag{3.99}$$

with

$$V_i = (-1)^{(N/2)+\mathrm{i}} \sum_{k=(i+1)/2}^{\min(i,N/2)} [k^{N/2}(2k!)]/[(N/2-k)!k!(k-1)!(i-k)!(2k-i)!] \tag{3.100}$$

where ln2 = 0.693147180559945, $t$ is the elapsed time, $F[i(\ln/2)/t)]$ is the appropriate groundwater flow Laplace transform equation in which $i(\ln/2)/t$ is substituted for the Laplace transform parameter $p$, $N$ is the even number of Stehfest terms (4, 6, 8, etc.), ! is a factorial, and $k$ is computed using integer arithmetic.

Type curve dimensionless drawdowns can be calculated at any dimensionless time by substituting $r^2S/Tt$ or $Tt/r^2S$ for $t$ in Equation 3.99.

When too few or too many Stehfest terms are used, dimensionless drawdowns will be highly irregular (sometimes spikes will appear) especially in early time solutions. In this case, the number of Stehfest terms is changed and dimensionless drawdowns are recalculated. The sum of $V_i$ values for any particular value of $N$ is 0. Tables of $V_i$ are presented by Walton (1996, pp. 62–64). Computer programs for calculating values of $V_i$ are presented by Dougherty (1989, pp. 564–569), Moench (1994), Barlow and Moench (1999), and Cheng (2000). Since $V_i$ depends only on $N$, it only needs to be calculated once for any value of $N$. The optimal value of $N$ is 8 for pumping test and slugged well equations and 4 for derivative slugged well equations and slug observation well equations, assuming a personal computer and double precision calculations are used. Values of $V_i$ for $N = 4$, $N = 6$, $N = 8$, and $N = 10$ are as follows:

$N = 4$

$V_i(1) = -0.2000000000000000D + 01$

$V_i(2) = 0.2600000000000000D + 02$

$V_i(3) = -0.4800000000000000D + 02$

$V_i(4) = 0.2400000000000000D + 02$

$N = 6$

$V_i(1) = 0.1000000000000000D + 01$

$V_i(2) = -0.4900000000000000D + 02$

$V_i(3) = \ \ 0.3660000000000000D + 03$
$V_i(4) = -0.8580000000000000D + 03$
$V_i(5) = \ \ 0.8100000000000000D + 03$
$V_i(6) = -0.2700000000000000D + 03$

$N = 8$
$V_i(1) = -0.3333333333333333D + 00$
$V_i(2) = \ \ 0.4833333333333333D + 02$
$V_i(3) = -0.9060000000000000D + 03$
$V_i(4) = \ \ 0.5464666666666667D + 04$
$V_i(5) = -0.1437666666666667D + 05$
$V_i(6) = \ \ 0.1873000000000000D + 05$
$V_i(7) = -0.1194666666666667D + 05$
$V_i(8) = \ \ 0.2986666666666667D + 04$

$N = 10$
$V_i(1) = \ \ 0.8333333333333333D - 01$
$V_i(2) = -0.3208333333333334D + 02$
$V_i(3) = \ \ 0.1279000000000000D + 04$
$V_i(4) = -0.1562366666666667D + 05$
$V_i(5) = \ \ 0.8424416666666666D + 05$
$V_i(6) = -0.2369575000000000D + 06$
$V_i(7) = \ \ 0.3759116666666667D + 06$
$V_i(8) = -0.3400716666666667D + 06$
$V_i(9) = \ \ 0.1640625000000000D + 06$
$V_i(10) = -0.3281250000000000D + 05$

## Confined Aquifer Laplace Transform Equations

Laplace transform dimensionless drawdown equations for confined aquifers with specified aquifer boundary and well conditions are as follows (Moench and Ogata, 1984, pp. 153–168; Johns et al., 1992, p. 73; Cheng, 2000; Moench et al., 2001):

Infinite aquifer with fully penetrating wells without pumped and observation well delayed response (wellbore storage) and pumped wellbore skin (Theis equation):

$$\overline{h}_D = K_0(rk^{1/2})/p \tag{3.101}$$

where $rk^{1/2}$ is a factor (discussed below Equation 3.115) and $p$ is the Laplace transform parameter.

Infinite aquifer with fully penetrating wells and pumped wellbore storage without observation well delayed response and pumped wellbore skin:

$$\overline{h}_D = K_0(rk^{1/2})/(p\ F_{wbs}) \tag{3.102}$$

$$F_{wbs} = W_D[(rk^{1/2})^2/r^2_D]K_0(rk^{1/2}/r_D) + (rk^{1/2}/r_D)K_1(rk^{1/2}/r_D) \tag{3.103}$$

where

$$W_D = (r_c^2 - r_p^2)/(2r_w^2S) \tag{3.104}$$

$$r_D = r/r_w \tag{3.105}$$

$r_c$ is the pumped well casing radius, $r_p$ is the pump pipe radius, $r_w$ is the pumped well effective radius, and $S$ is the aquifer storativity.

Values of $K_1(x)$ are calculated with the following polynomial approximations presented by Abramowitz and Stegun (1964):

when $0 < x \leq 2$

$$K_1(x) = a_{10} + a_{11}[a_1 + (x^2/4)a_{12}] \tag{3.106}$$

$a_1 = 1.00000000$ $a_2 = 0.15443144$
$a_3 = -0.67278579$ $a_4 = 0.18156897$
$a_5 = -0.01919402$ $a_6 = 0.00110404$
$a_7 = -0.00004686$ $a_8 = a_6 + (x^2/4)a_7$
$a_9 = a_4 + (x^2/4)[a_5 + (x^2/4)a_8]$
$a_{10} = \log(x/2)I_1(x)$
$a_{11} = 1/x$
$a_{12} = \{a_2 + (x^2/4)[a_3 + (x^2/4)a_9]\}$

with abs($x$) < 3.75

$$I_1(x) = x\{a_1 + (x/3.75)^2[a_2 + (x/3.75)^2a_{10}]\} \tag{3.107}$$

$a_1 = 0.5$ $a_2 = 0.87890594$
$a_3 = 0.51498869$ $a_4 = 0.15084934$
$a_5 = 0.02658733$ $a_6 = 0.00301532$
$a_7 = 0.00032411$
$a_8 = a_6 + (x/3.75)^2a_7$
$a_9 = a_4 + (x/3.75)^2[a_5 + (x/3.75)^2a_8]$
$a_{10} = a_3 + (x/3.75)^2a_9$

with abs($x$) ≥ 3.75

$$I_1(x) = [\exp(b_1)/(b_1)^{1/2}]a_{13} \tag{3.107a}$$

$a_1 = 0.39894228$ $a_2 = 0.03988024$
$a_3 = -0.00362018$ $a_4 = 0.00163801$
$a_5 = -0.01031555$ $a_6 = 0.02282967$

$$a_7 = -0.02895312 \qquad a_8 = 0.01787654$$
$$a_9 = -0.00420059$$
$$a_{10} = a_6 + b_2[a_7 + b_2(a_8 + b_2a_9)]$$
$$a_{11} = a_5 + (b_2a_{10})$$
$$a_{12} = a_4 + (b_2a_{11})$$
$$a_{13} = a_1 + b_2[a_2 + b_2(a_3 + b_2a_{12})]$$
$$b_1 = \text{abs}(x)$$
$$b_2 = 3.75/b_1$$

when $x > 2$

$$K_1(x) = [a_1 + (2/x)a_{10}][\exp(-x)/x^{0.5}] \tag{3.108}$$

$$a_1 = 1.25331414 \qquad a_2 = 0.23498619$$
$$a_3 = -0.0365562 \qquad a_4 = 0.01504268$$
$$a_5 = -0.00780353 \qquad a_6 = 0.00325614$$
$$a_7 = -0.00068245$$
$$a_8 = a_6 + (2/x)a_7$$
$$a_9 = a_4 + (2/x)[a_5 + (2/x)a_8]$$
$$a_{10} = a_2 + (2/x)[a_3 + (2/x)a_7]$$

Infinite aquifer with partially penetrating wells and pumped wellbore storage without observation well delayed response and pumped wellbore skin:

$$\overline{h}_D = [K_0(rk^{1/2}) + F_{pp}]/[p\ (F_{wbs})] \tag{3.109}$$

$$F_{pp} = 2/[(x_L - x_D)(x'_L - x'_D)]\sum_{n}^{\infty} 1/n^2\ [\sin(x_Ln) - \sin(x_Dn)]$$

$$[\sin(x'_Ln) - \sin(x'_Dn)][K_0[(rk^{1/2})^2 + (K_v/K_h)(n\pi r/b)^2]^{0.5} \tag{3.110}$$

where

$$x_L = \pi L/b \tag{3.111}$$

$$x_D = \pi D/b \tag{3.112}$$

$$x'_L = \pi L'/b \tag{3.113}$$

$$x'_D = \pi D'/b \tag{3.114}$$

$K_v$ is the aquifer vertical hydraulic conductivity, $K_h$ is the aquifer horizontal hydraulic conductivity, $b$ is the aquifer thickness, $L$ is the depth from the aquifer top to the pumped well base, $D$ is the depth from the aquifer top to the top of the pumped well screen, $L'$ is the depth from the aquifer top to the observation well base, $D'$ is the depth from the aquifer top to the top of the observation well screen.

Finite aquifer with partially penetrating wells and pumped wellbore storage without observation well delayed response and pumped wellbore skin:

$$\overline{h}_D = [K_0(rk^{1/2}) + F_{pp} + K_0(r_{iw}k^{1/2})]/(p\ F_{wbs}) \tag{3.115}$$

The $rk^{1/2}$ factor is the Laplace–domain transform solution for a particular set of aquifer and real well conditions. The $r_{iw}k^{1/2}$ factor is the Laplace–domain transform solution for a particular set of aquifer and image well conditions. $K_0(\ldots)$ is the modified Bessel function of second kind and order zero. The Bessel function associated with a boundary image well is subtracted when there is a recharge boundary or is added when there is a barrier boundary. Additional Bessel functions associated with boundary image wells are added when there are several image wells. The effect of image well partially penetration is assumed to be negligible because the distance between the observation well and image well is usually large. The effects of boundary wellbore storage are assumed to be appreciable.

Finite aquifer with partially penetrating wells and pumped wellbore storage without observation well delayed response and with pumped wellbore skin:

$$\overline{h}_D = [K_0(rk^{1/2}) + (rk^{1/2})S_{wsf}K_1(rk^{1/2}) + F_{pp} + K_0(r_{iw}k^{1/2})]/(p\ F_{wbss}) \tag{3.116}$$

where

$$S_{wsf} = K_h d_s/(K_v\, r_w) \tag{3.117}$$

$S_{wsf}$ is the wellbore skin factor, $K_h$ is the aquifer horizontal hydraulic conductivity, $K_v$ is the aquifer vertical hydraulic conductivity, $r_w$ is the pumped well effective radius, and $d_s$ is the skin thickness.

Finite aquifer with partially penetrating wells, pumped wellbore storage, observation well delayed response, and pumped wellbore skin (Moench et al., 2001, p. 12):

$$\overline{h}_{mD} = \overline{h}_D/(1 + W_{dp}p) \tag{3.118}$$

where $\overline{h}_{mD}$ is the Laplace transform dimensionless drawdown with observation well delayed response and $W_{dp}$ is a dimensionless parameter defined as

$$W_{dp} = \pi r_o^2/(2\pi r_w^2 S_s F') \tag{3.119}$$

$r_o$ is the observation well casing radius, $r_w$ is the pumped well effective radius, $S_s = S/b$, and

$$F' = L_s/[\ln(x + 1 + x^2)^{0.5}] \tag{3.120}$$

$$x = (K_h/K_v)^{0.5} L_s/(2r_o) \tag{3.121}$$

$K_h$ is the aquifer horizontal hydraulic conductivity, $K_v$ is the aquifer vertical hydraulic conductivity, $F'$ is a shape factor defined by Hvorslev (1951, case 8), $L_s$ is the observation well or piezometer screen length, and $r_o$ is the observation well radius. Delayed response in an observation piezometer cannot be simulated with Equation 3.119 because the screened length of the observation piezometer is 0 (Barlow and Moench, 1999, p. 12). Drawdown in an observation piezometer without delayed response is calculated based on the depth to the piezometer center. Drawdown in an observation piezometer with delayed response is simulated by assuming the observation piezometer is a partially penetrating observation well with a very short (1 ft) screen.

The $rk^{1/2}$ factors for commonly encountered confined aquifer conditions are:

Confined nonleaky aquifer (Moench and Ogata, 1984, p. 153):

$$rk^{1/2} = (r^2Sp/T)^{0.5} \tag{3.122}$$

where $r$ is the distance from the pumped well to the observation well, $S$ is the aquifer storativity, $p$ is the Laplace–domain variable, and $T$ is the aquifer transmissivity.

Confined leaky aquifer without confining unit storativity and a source unit above the confining unit (Hantush, 1964, pp. 331–332):

$$rk^{1/2} = [r^2Sp/T + (r/B)^2]^{0.5} \tag{3.123}$$

where

$$B = [T/(K'/b')]^{0.5} \tag{3.124}$$

$r$ is the distance from the pumped well to the observation well, $S$ is the aquifer storativity, $p$ is the Laplace–domain variable, $T$ is the aquifer transmissivity, $K'$ is the confining unit vertical hydraulic conductivity, and $b'$ is the confining unit thickness.

Confined leaky aquifer with confining unit storativity and the confining unit overlain by a source unit (Moench and Ogata, 1984, pp. 153–154):

$$rk^{1/2} = \{r^2Sp/T + 4(r^2Sp/T)^{0.5}\beta\coth[4(r^2Sp/T)^{0.5}\beta/(r/B)^2]\}^{0.5} \tag{3.125}$$

where

$$\beta = [K'r/(4K_hb)][TS'/(SK'b')]^{0.5} \tag{3.126}$$

$$r/B = (r/b)[K'b/(K_hb')]^{0.5} \tag{3.127}$$

$r$ is the distance from the pumped well to the observation well, $S$ is the aquifer storativity, $p$ is the Laplace–domain variable, $T$ is the aquifer transmissivity, $K_h$ is the aquifer horizontal hydraulic conductivity, $b$ is the aquifer thickness, $K'$ is the confining unit vertical hydraulic conductivity, $b'$ is the confining unit thickness, and $S'$ is the confining unit storativity.

Confined leaky aquifer with confining unit storativity and the confining unit overlain by an impermeable unit (Hantush, 1964, pp. 331–332):

$$rk^{1/2} = \{r^2Sp/T + 4(r^2Sp/T)^{0.5}\beta\tanh[4(r^2Sp/T)^{0.5}\beta/(r/B)^2]\}^{0.5} \quad (3.128)$$

where

$$\beta = [K'r/(4K_hb)][TS'/(SK'b')]^{0.5} \quad (3.129)$$

$$r/B = (r/b)[K'b/(K_hb')]^{0.5} \quad (3.130)$$

$r$ is the distance from the pumped well to the observation well, $S$ is the aquifer storativity, $p$ is the Laplace–domain variable, $T$ is the aquifer transmissivity, $K'$ is the confining unit vertical hydraulic conductivity, $K_h$ is the aquifer horizontal hydraulic conductivity, $b$ is the aquifer thickness, $b'$ is the confining unit thickness, and $S'$ is the confining unit storativity.

Confined leaky aquifer with confining unit storativity and the confining unit contains the water table (Cooley and Case, 1973):

$$\begin{aligned} rk^{1/2} = \{&r^2Sp/T + 4(r^2Sp/T)^{0.5}\beta\tanh[4(r^2Sp/T)^{0.5}\beta/(r/b)^2] \\ &+ (r^2Sp/T)\mathrm{sech}^2[4(r^2Sp/T)^{0.5}\beta/(r/B)^2]/[(r^2Sp/T)(L_c/b)/(r/B)^2 \\ &+ (r/B)^2S/(16\beta^2S_{cY}) + (r^2Sp/T)/(4\beta)\tanh\,(4r^2Sp/T)\beta/(r/B)^2]\}^{0.5} \end{aligned} \quad (3.131)$$

where

$$\beta = [K'r/(4K_hb)][TS'/(SK'b')]^{0.5} \quad (3.132)$$

$$r/B = (r/b)\{K'b/[K_h(b' + L_{cf})]\}^{0.5} \quad (3.133)$$

$r$ is the distance from the pumped well to the observation well, $S$ is the aquifer storativity, $p$ is the Laplace–domain variable, $T$ is the aquifer transmissivity, $b$ is the aquifer thickness, $K'$ is the confining unit vertical hydraulic conductivity, $K_h$ is the aquifer horizontal hydraulic conductivity, $b'$ is the confining unit thickness, $S'$ is the confining unit storativity, $S_{cY}$ is the confining unit specific yield, and $L_{cf}$ is the capillary fringe thickness.

Laplace transform dimensionless equations for an infinite confined leaky aquifer with fully penetrating wells without wellbore storage and skin, confining unit storativity, and a confining unit overlain by a variable head aquifer (two-aquifer system with drawdown in the unpumped aquifer) are as follows (Neuman and Witherspoon, 1969a, 1969b; see Moench and Ogata, 1984):

For the pumped aquifer

$$\bar{h}_D = 1/p\ [(A_2 - k_1)/D_{vh}]\ K_0(rk_1^{1/2}) - 1/p[(A_2 - k_2)/D_{vh}]\ K_0(rk_2^{1/2}) \quad (3.134)$$

where

$$k_1 = 1/2[A_1 + A_2 - D_{vh}] \quad (3.135)$$

$$k_2 = 1/2[A_1 + A_2 + D_{vh}] \quad (3.136)$$

$$D_{vh} = [4B_1B_2 + (A_1 - A_2)^2]^{0.5} \quad (3.137)$$

$$A_1 = r^{-2}[\eta^2 + 4\eta\beta_{11}\coth(\psi)] \quad (3.138)$$

$$A_2 = r^{-2}[\eta^2(\alpha_1/\alpha_2) + 4\eta\beta_{21}\ (\alpha_1/\alpha_2)^{0.5}\coth(\psi)] \quad (3.139)$$

$$B_1 = r^{-2}\ 4\eta\beta_{11}\ [1/\sinh(\psi)] \quad (3.140)$$

$$B_2 = r^{-2}\ 4\eta\beta_{21}\ (\alpha_1/\alpha_2)^{0.5}\ [1/\sinh(\psi)] \quad (3.141)$$

$$\psi = 4\eta\beta_{11}\ (r/B_{11})^{-2} \quad (3.142)$$

$$\beta_{11} = 1/4\ K'/K_1\ r/b_1\ (\alpha_1/\alpha')^{0.5} \quad (3.143)$$

$$r/B_{11} = r/b_1\ (K'/K_1\ b_1/b')^{0.5} \quad (3.144)$$

$$\beta_{21} = \beta_{11}\ T_1/T_2\ (\alpha_2/\alpha_1)^{0.5} \quad (3.145)$$

$$r/B_{21} = r/B_{11}\ (T_1/T_2)^{0.5} \quad (3.146)$$

$$\alpha_1 = T_1/S_1 \quad (3.147)$$

$$\alpha_2 = T_2/S_2 \quad (3.148)$$

$$\alpha' = K'b'/S' \quad (3.149)$$

$$\eta = [(r^2S_1/T_1)\ p]^{0.5} \quad (3.150)$$

For the unpumped aquifer

$$\bar{h}_{D2} = [B_2/(pD_{vh})]\ [K_0(rk_1^{1/2}) - K_0(rk_2^{1/2})] \quad (3.151)$$

For the confining unit

$$\bar{h}'_D = [\sinh(\psi z/b')/\sinh(\psi)]\,\bar{h}_{D2} + \{\sinh[\psi(1 - z/b')]/\sinh(\psi)\}\bar{h}_{D1} \quad (3.152)$$

For two aquifers with identical hydraulic properties

$$\bar{h}_{D1} = 1/(2p)\ [K_0(rk_1^{1/2}) + K_0(rk_2^{1/2})] \quad (3.153)$$

where $p$ is the Laplace–domain variable, $r$ is the distance from the pumped well to the observation well, $b_1$ is the pumped aquifer thickness, $b'$ is the confining unit thickness, $K_1$ is the pumped aquifer horizontal hydraulic conductivity, $K'$ is the confining unit vertical hydraulic conductivity, $r$ is the distance from the pumped well to the observation well, $T_1$ is the pumped aquifer transmissivity, $T_2$ is the unpumped aquifer transmissivity, $S_1$ is the pumped aquifer storativity, $S_2$ is the unpumped aquifer storativity or specific yield, and $S'$ is the confining unit storativity.

The effects of drawdown in the unpumped aquifer may not be appreciable during the short duration of most aquifer tests, but, these effects can be quite significant over longer periods of time.

Dennis and Motz (1998) extended the Neuman and Witherspoon (1969b) two-aquifer system equations to cover pumping as well as reduction in evapotranspiration from the aquifer above the confining unit. These equations are included in their Fortran program NSSCON. Cheng (2000) extended the Neuman and Witherspoon (1969b) two-aquifer system equations to cover multiple-aquifer-confining unit systems. These equations are included in a suite of Fortran programs. Both the Dennis and Motz and Cheng equations are for infinite aquifers and fully penetrating wells with no wellbore storage and skin.

Moench (1985) extended the Hantush (1960) theory of a confined leaky aquifer overlain and underlain by confining units to cover wellbore storage and skin. The upper boundary of the overlying confining unit and the lower boundary of the underlying confining unit can be constant head or no-flow boundaries.

Confined fissure and block aquifer (double porosity) wells are fully penetrating. The fissure has a skin; the block is leaky with storativity and is overlain by a source unit (Moench, 1984, pp. 831–846).

For a slab-shaped block:

$$rk^{1/2} = \{(r^2Sp/T) + r_{Df}^{\,2}(m_f)[\tanh(m_f)]/[1 + S_f(m_f)\tanh(m_f)]\}^{0.5} \quad (3.154)$$

For a sphere-shaped block:

$$rk^{1/2} = \{(r^2Sp/T) + 3r_{Df}^{\,2}[(m_f)\coth(m_f) - 1]/\{1 + S_f[m_f\coth(m_f) - 1]\}^{0.5} \quad (3.155)$$

where

$$r_{Df} = [r/(b'_b/2)][K'_b/K_f]^{0.5} \quad (3.156)$$

$$m_f = (S_r p)^{0.5}/r_{Df} \tag{3.157}$$

$$S_r = S'_b/S_s \tag{3.158}$$

$$S_f = K'_b b_s/[K_f(b'_b/2)] \tag{3.159}$$

$T$ is the fissure transmissivity, $K_f$ is the fissure horizontal hydraulic conductivity, $K'_f$ is the block vertical hydraulic conductivity, $b'_b$ is the average block thickness between fissure zones, $S'_b$ is the block specific storage, $S_f$ is the fissure specific storage, $S$ is the aquifer storativity, $p$ is the Laplace–domain variable, $K_s$ is the fissure skin hydraulic conductivity, $r$ is the distance from the pumped well to the observation well, $b_s$ is the fissure skin thickness. The aquifer is assumed to consist of two interacting, overlapping continua: a continuum of low-hydraulic conductivity, primary porosity blocks and a continuum of high-hydraulic conductivity, secondary porosity fissures. The parameters of the continuums are homogeneous and isotropic. The double-porosity aquifer is confined above and below by impermeable formations. Groundwater enters a single pumped wellbore through the fissures and not the block. The discharge rate is constant. There is no initial groundwater flow. There is transient flow from blocks to fissures causing type curves to show a transition from early to late time. The length of the transition time is controlled by $S_r$ and the vertical position of the transient period is controlled by $r_D$. Slab-shaped blocks are usually assumed. Closely spaced water entries are needed to justify the use of sphere-shaped blocks. $S_f$ is typically 1/10 to 1/100 of $S'_b$. $K_f$ is typically 0.001 to 10 ft/day and $K'_b$ is typically 1E-7 to 1E-4 ft/day. $T$, $S'_b$, and $S_f$ are associated with the combined fissure and block thickness.

Note that the $rk^{1/2}$ factors for leaky and fissure and block aquifers are equal to the $rk^{1/2}$ factor for the nonleaky aquifer plus source terms.

## Unconfined Aquifer Laplace Transform Equations

Laplace transform dimensionless drawdown equations for unconfined aquifers with specified aquifer and well conditions are as follows (Moench, 1997, 1998; Moench et al., 2001 and Addendum):

For a pumped well:

$$\bar{h}_D = 2(A + S_{wsf})/\{p(1_D - d_D)[1 + W_d p(A + S_{wsf})]\} \tag{3.160}$$

where

$$A = 2/(l_D - d_D)\sum_{n=0}^{\infty} K_0(q_n)\{\sin[n\pi(1 - d_D)] - \sin[n\pi(1 - l_D)]\}^2/ \{n\pi q_n K_1(q_n)\} \tag{3.161}$$

$$l_D = L/b \tag{3.162}$$

$$d_D = D/b \tag{3.163}$$

$$W_d = \pi r_{ce}^2/[2\pi r_w^2 S_s(L - D)] \tag{3.164}$$

$$q_n = (\varepsilon_n^2 \beta_w + p)^{0.5} \tag{3.165}$$

$\varepsilon_n$, where $n = 0, 1, 2, \ldots$ are the roots of

$$\varepsilon_n \tan(\varepsilon_n) = p/M \sum_{n=1}^{M} [1/(\sigma\beta_w + p/\gamma_m)] \tag{3.166}$$

$M$ is the number of empirical constants for gradual drainage from the unsaturated zone, $\sigma = S/S_y$, and $S$ is the aquifer storativity and $S_y$ is the aquifer specific yield:

$$\gamma_m = \alpha_m b S_y/K_v \tag{3.167}$$

$\alpha_m$ is the $m$th empirical constant for gradual drainage from the unsaturated zone.

$$\beta_w = K_D r_{wD}^2 \tag{3.168}$$

$$K_D = K_v/K_h \tag{3.169}$$

$$r_{wD} = r_w/b \tag{3.170}$$

$$\beta = \beta_w r_D^2 \tag{3.171}$$

$$r_D = r/r_w \tag{3.172}$$

$$r_{ce} = (r_c^2 - r_p^2)^{0.5} \tag{3.173}$$

$S_{wsf}$ is the wellbore skin factor $= K_h d_s/(K_v r_w)$, $K_h$ is the aquifer horizontal hydraulic conductivity, $d_s$ is the skin thickness (for simplicity, drawdown due to skin is presumed to increase linearly with the discharge rate, Tien-Chang Lee, 1999, p. 181), $K_v$ is the aquifer vertical hydraulic conductivity, $r_w$ is the pumped well effective radius, $S_s$ is the aquifer specific storativity, $b$ is the aquifer thickness, $r$ is the distance from the pumped well to the observation well, $r_c$ is the pumped well casing radius, $r_p$ is the pump pipe radius, $p$ is the Laplace–domain variable, $L$ is the depth from the aquifer top to the pumped well base, and $D$ is the depth from the aquifer top to the top of the pumped well screen.

For an observation well:

$$\overline{h}_D = 2E/\{p(l_D - d_D)[1 + W_d p(A + S_{wsf})]\} \tag{3.174}$$

where

$$E = 2\sum_{n=0}^{\infty} K_0(q_n r_D)\{\sin[n\pi(1 - d_D)] - \sin[n\pi(1 - l_D)]\}/$$
$$\{n\pi q_n K_1(q_n)[n\pi + 0.5\sin(2n\pi)]\}$$
$$[\sin(n\pi z_{D2}) - \sin(n\pi z_{D1})]/(z_{D2} - z_{D1}) \qquad (3.175)$$

$$l_D = L/b \qquad (3.176)$$

$$d_D = D/b \qquad (3.177)$$

$$z_{D1} = z_1/b \qquad (3.178)$$

$$z_{D2} = z_2/b \qquad (3.179)$$

$$W_d = \pi r_{ce}^2/[2\pi r_w^2 S_s(L - D)] \qquad (3.180)$$

$$q_n = (\varepsilon_n^2\beta_w + p)^{0.5} \qquad (3.181)$$

$\varepsilon_n$, where $n = 0, 1, 2, \ldots$ are the roots of

$$\varepsilon_n \tan(\varepsilon_n) = p/M\sum_{n=1}^{M} [1/(\sigma\beta_w + p/\gamma_m)] \qquad (3.182)$$

$M$ is the number of empirical constants for gradual drainage from the unsaturated zone, $\sigma = S/S_y$, and $S$ is the aquifer storativity and $S_y$ is the aquifer specific yield:

$$\gamma_m = \alpha_m b S_y/K_v \qquad (3.183)$$

$\alpha_m$ is the $m$th empirical constant for gradual drainage from the unsaturated zone.

$$\beta_w = K_D r_{wD}^2 \qquad (3.184)$$

$$K_D = K_v/K_h \qquad (3.185)$$

$$r_{wD} = r_w/b \qquad (3.186)$$

$$q_n r_D = (\varepsilon_n^2\beta + p r_D^2)^{0.5} \qquad (3.187)$$

$$\beta = \beta_w r_D^2 \qquad (3.188)$$

$$r_D = r/r_w \qquad (3.189)$$

$$r_{ce} = (r_c^2 - r_p^2)^{0.5} \qquad (3.190)$$

$S_{wsf}$ is the wellbore skin factor = $K_h d_s/(K_v r_w)$, $K_h$ is the aquifer horizontal hydraulic conductivity, $d_s$ is the skin thickness (for simplicity, drawdown due to skin is presumed to increase linearly with the discharge rate (Tien-Chang Lee, 1999, p. 181), $K_v$ is the aquifer vertical hydraulic conductivity, $r_w$ is the pumped well effective radius, $S_s$ is the aquifer specific storativity, $b$ is the aquifer thickness, $r$ is the distance from the pumped well to the observation well, $r_c$ is the pumped well casing radius, $r_p$ is the pump pipe radius, $p$ is the Laplace–domain variable, 1 is the depth from the aquifer top to the pumped well base, $d$ is the depth from the aquifer top to the top of the pumped well screen, $z_2$ is the depth from the aquifer top to the observation well base, and $z_1$ is the depth from the aquifer top to the top of the observation well screen.

For a piezometer:

$$\bar{h}_D = 2E/\{p(l_D - d_D)[1 + W_d p(A + S_{wsf})]\} \tag{3.191}$$

where

$$E = 2\sum_{n=0}^{\infty} K_0(q_n r_D)\{\sin[n\pi(1 - d_D)] - \sin[n\pi(1 - l_D)]\}/$$
$$[n\pi q_n K_1(q_n)] \cos(n\pi z_D) \tag{3.192}$$

$$l_D = Ll/b \tag{3.193}$$

$$d_D = D/b \tag{3.194}$$

$$z_D = z_p/b \tag{3.195}$$

$$q_n = (\varepsilon^2{}_n\beta_w + p)^{0.5} \tag{3.196}$$

$$q_n r_D = (\varepsilon^2{}_n\beta + p r_D{}^2)^{0.5} \tag{3.197}$$

$\varepsilon_n$, where $n = 0, 1, 2, \ldots$ are the roots of

$$\varepsilon_n \tan(\varepsilon_n) = p/M \sum_{n=1}^{M} [1/(\sigma\beta_w + p/\gamma_m)] \tag{3.198}$$

$M$ is the number of empirical constants for gradual drainage from the unsaturated zone, $\sigma = S/S_y$, and $S$ is the storativity and $S_y$ is the specific yield:

$$\gamma_m = \alpha_m b S_y/K_v \tag{3.199}$$

$\alpha_m$ is the $m$th empirical constant for gradual drainage from the unsaturated zone.

$$W_d = \pi r_{ce}^2/[2\pi r_w^2 S_s(L - D)] \tag{3.200}$$

$$\beta_w = K_D r_{wD}^2 \tag{3.201}$$

$$K_D = K_v / K_h \tag{3.202}$$

$$r_{wD} = r_w/b \tag{3.203}$$

$$\beta = \beta_w r_D^2 \tag{3.204}$$

$$r_D = r/r_w \tag{3.205}$$

$$r_{ce} = (r_c^2 - r_p^2)^{0.5} \tag{3.206}$$

$S_{wsf}$ is the wellbore skin factor = $K_h d_s/(K_v r_w)$, $K_h$ is the aquifer horizontal hydraulic conductivity, $d_s$ is the skin thickness (for simplicity, drawdown due to skin is presumed to increase linearly with the discharge rate (Tien-Chang Lee, 1999, p. 181), $K_v$ is the aquifer vertical hydraulic conductivity, $r_w$ is the pumped well effective radius, $S_s$ is the aquifer specific storativity, $b$ is the aquifer thickness, $r$ is the distance from the pumped well to the observation well, $r_c$ is the pumped well casing radius, $r_p$ is the pump pipe radius, $p$ is the Laplace–domain variable, $L$ is the depth from the aquifer top to the pumped well base, and $D$ is the depth from the aquifer top to the top of the pumped well screen, and $z_p$ is the vertical distance above the base of the aquifer to the center of the piezometer screen. The process of gradual drainage from the unsaturated zone above the water table, effects of well partial penetration, pumped wellbore storage, and pumped well skin are simulated. Well loss due to the turbulent flow near the pumped well is not simulated.

## Induced Streambed Infiltration Fourier–Laplace Transform Equations

Induced streambed infiltration Fourier–Laplace transform dimensionless drawdown equations for unconfined and confined leaky aquifers are presented by Butler et al. (2001); Butler and Tsou (2001); and Zhan and Butler (2005). These equations assume negligible wellbore storage and skin, fully penetrating wells, and finite width partially penetrating streambeds. Equations for unconfined aquifers (Butler and Tsou, 2001) are as follow:

Beyond the streambed in Zone 1:

$$\bar{\bar{\Phi}}_1\,(\varepsilon, \omega, p) = (T_f)[e^a + e^b] \tag{3.207}$$

Beneath the streambed in Zone 2:

$$\overline{\overline{\Phi}}_2\ (\varepsilon, \omega, p) = (T_f)[(A_1)e^c + (B_1)e^d] \tag{3.208}$$

Between pumped well and streambed in Zone 3:

$$\overline{\overline{\Phi}}_3\ (\varepsilon, \omega, p) = (T_f)[(D_1)e^f + (E_1)e^g],\ 0 \le \varepsilon \le \alpha \tag{3.209}$$

Between pumped well and right boundary in Zone 3:

$$\overline{\overline{\Phi}}_3\ (\varepsilon, \omega, p) = [(T_f)(G_1)/(H_1)][e^f + e^h],\ \alpha < \varepsilon \le X_{RB} \tag{3.210}$$

where

$$\Phi_i \text{ (dimensionless drawdown)} = s_i T_3/Q,\ i = 1, 3 \tag{3.211}$$

$$\tau \text{ (dimensionless time)} = (T_3 t)/(w^2 S_3) \tag{3.212}$$

$$\xi = x/w \tag{3.213}$$

$$\eta = y/w \tag{3.214}$$

$$\alpha = a/w \tag{3.215}$$

$$B \text{ (stream leakance)} = (k'w^2)/(b'T_2) \tag{3.216}$$

$$X_{RB} = x_{rb}/w \tag{3.217}$$

$$X_{LB} = x_{lb}/w \tag{3.218}$$

$$\gamma_i = T_{i+1}/T_i\ i = 1, 2 \tag{3.219}$$

$$P_i = \mu_i/\mu_3\ i = 1, 2 \tag{3.220}$$

$$\mu_i = S_i/T_i\ i = 1, 3 \tag{3.221}$$

$$\lambda_1 = (\omega^2 + P_1 p)^{0.5} \tag{3.222}$$

$$\lambda_2 = (\omega^2 + B + P_2 p)^{0.5} \tag{3.223}$$

$$\lambda_3 = (\omega^2 + p)^{0.5} \tag{3.224}$$

$$A_1 = 1/2(e^r + e^{r1}) + [\lambda_1/(/(2\ \gamma_1\ \lambda_2)][e^r - e^{r1}] \tag{3.225}$$

$$B_1 = 1/2(e^{r2} + e^{r3}) - [\lambda_1/(/(2\ \gamma_1\ \lambda_2)]\ [e^{r2} - e^{r4}] \tag{3.226}$$

$$D_1 = 1/2[(A_1) + (B_1)] + [\lambda_1/(2\,\gamma_2\,\lambda_3)]\ [(A_1) - (B_1)] \tag{3.227}$$

$$a = 2\lambda_1 X_{\Lambda B} + \lambda_1\xi \tag{3.228}$$

$$b = -\lambda_1\xi \tag{3.229}$$

$$c = \lambda_2\xi \tag{3.230}$$

$$d = -\lambda_2\xi \tag{3.231}$$

$$f = \lambda_3\xi \tag{3.232}$$

$$g = -\lambda_3\xi \tag{3.233}$$

$$h = 2\ \lambda_3 X_{RB} - \lambda_3\xi \tag{3.234}$$

$$r = 2\ \lambda_1 X_{LB} - \lambda_1 + \lambda_2 \tag{3.235}$$

$$r1 = \lambda_1 + \lambda_2 \tag{3.236}$$

$$r2 = 2\ \lambda_1 X_{LB} - \lambda_1 - \lambda_2 \tag{3.237}$$

$$r3 = \lambda_1 - \lambda_2 \tag{3.238}$$

$$r4 = -\lambda_1 - \lambda_2 \tag{3.239}$$

$$E_1 = 1/2[(A_1) + (B_1)] - [\lambda_2/(2\ \gamma_2\ \gamma_3)][(A_1) - (B_1)] \tag{3.240}$$

$$F_1 = -1/[\ \lambda_3 p(2\pi)^{0.5}] \tag{3.241}$$

$$G_1 = (D_1)e^{r5} + (E_1)e^{r6} \tag{3.242}$$

$$r5 = \lambda_3\alpha \tag{3.243}$$

$$r6 = -\lambda_3\alpha \tag{3.244}$$

$$H_1 = e^{r5} + e^{r7} \tag{3.245}$$

$$r7 = 2\lambda_3 X_{RB} - \lambda_3\alpha \tag{3.246}$$

$$J_1 = (D_1)e^{r5} - (E_1)e^{-r5} \tag{3.247}$$

$$K_1 = e^{r5} - e^{r7} \tag{3.248}$$

$$T_f = (F_1)(H_1)/[(G_1)(K_1) - (J_1)(H_1)] \tag{3.249}$$

$\overline{\overline{\Phi}}_i$ = Fourier–Laplace transform of $\Phi_i$, $i = 1,3$; $p$ = Laplace–transform variable; $\omega$ = Fourier transform variable; $s_i$ is the drawdown in Zone $i$; $T_i$ is the aquifer transmissivity in Zone $i$; $Q$ is the discharge rate; $t$ is the elapsed time; $w$ is the streambed width; $S_i$ is the aquifer storativity in Zone $i$; $x$ is the $X$ coordinate; $y$ is the $Y$ coordinate; $a$ is the distance from the streambed to the pumped well; $k$ is the streambed vertical hydraulic conductivity; $b$ is the streambed thickness; $x_{rb}$ is the distance from the right boundary to the right side of the streambed; $x_{lb}$ is the distance from the left boundary to the right side of the streambed; the streambed bank nearest the pumped well is the zero $X$-coordinate baseline; $X$ coordinates are positive to the right of the baseline and negative to the left of the baseline; the line at a right angle to the streambed through the pumped well is the zero $Y$-coordinate baseline; $Y$ coordinates are positive above the baseline and negative below the baseline; parallel barrier boundaries occur to the right and left of the streambed baseline; distances to the barrier boundaries are positive to the right of the streambed baseline and negative to the left of the streambed baseline; the effects of the barrier boundaries become negligible when the distances from the streambed baseline to the boundaries are large enough (10,000 ft); Zone 1 refers to the aquifer left beyond the streambed from the pumped well; Zone 2 refers to the aquifer beneath the streambed; and Zone 3 refers to the aquifer to the right of the streambed. The equations given above are most readily evaluated using a numerical scheme. A Mathematica® add-on package (Mallet, 2000) can be used for the joint Fourier–Laplace numerical inversion.

The Laplace solution for stream depletion is (Butler and Tsou, 2001):

$$\Delta\overline{Q}(p) = B/(\gamma_2\lambda^*{}_2)(T_f)[(A_1)(1 - e^{-r8}) - (B_1)(1 - e^{r8})] \tag{3.250}$$

where

$$r8 = (B + P_2 p)^{0.5} \tag{3.251}$$

$$P_2 = (S_2/T_2)/(S_3/T_3) \tag{3.252}$$

Butler and Tsou (2000) developed the Fortran analytical program StrpStrm for calculating time-drawdown values with induced streambed infiltration in unconfined aquifers. The program can be downloaded at www.kgs.ku.edu/StreamAq/Software/strp.html.

Hunt (1999) presents Fourier–Laplace and other analytical equations for calculating time-drawdown and stream depletion values with induced streambed infiltration in unconfined aquifers. These equations assume negligible wellbore storage and skin, fully penetrating wells, and finite width partially penetrating streambeds.

Fox and Durnford (2002) developed the Fortran analytical program STRMAQ for calculating time-drawdown values with induced streambed infiltration in unconfined and confined nonleaky aquifers. STRMAQ uses analogous well functions to combine WTAQ (Barlow and Moench, 1999) and induced streambed infiltration equations presented by Hunt (1999). STRMAQ requires three files: filename, input, and output. STRMAQ accounts for well partial penetration, wellbore skin, wellbore storage, and finite width partially penetrating streambeds. STRMAQ can be downloaded at www.engr.colostate.edu/~durnford/projects/ NAPL/STRMAQ_Readme.txt.

## Wellbore Skin Effects

Effects of any well skin (see Figure 3.4) should be considered in estimating the pumped well effective radius (Moench et al., 2002, p. 18). If the well skin is less permeable than the aquifer, drawdown in the pumped well is increased (the effective radius decreases) and there is an apparent increase in wellbore storage, which reduces drawdowns in the aquifer at early times. If the well skin is more permeable than the aquifer, drawdown in the pumped well is decreased (the effective radius increases) and there is an apparent decrease in wellbore storage, which increases drawdowns in the aquifer at early times.

The pumped well effective radius can be estimated by calculating pumped well drawdowns for a selected time based on aquifer parameter values estimated with observation well data and several trial effective radius values. Calculated drawdowns are compared with the measured drawdown for the selected time. The trial effective radius that results in a match between calculated and measured drawdowns is assigned to the pumped well.

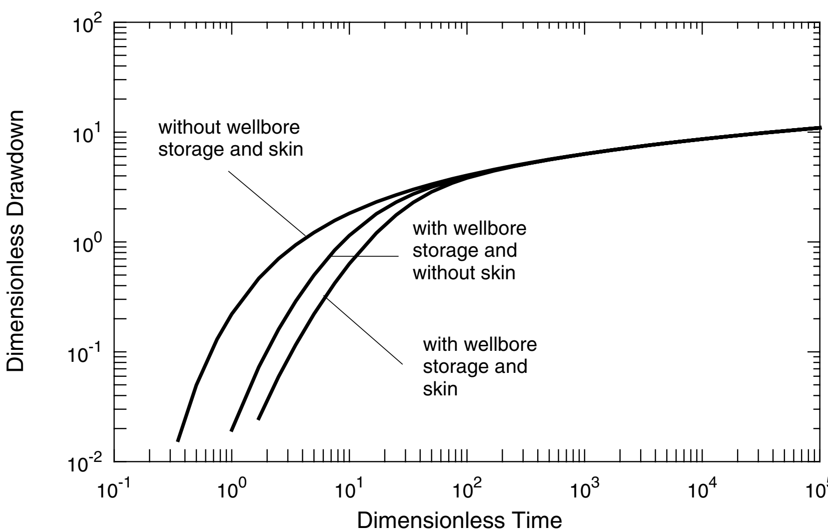



**FIGURE 3.4** Graph showing wellbore storage and skin effects on pumping test type curve values.

## STEHFEST ALGORITHM SLUG TEST MATHEMATICAL MODELING EQUATIONS

Several Stehfest algorithm slug test mathematical modeling equations are available (Dougherty, 1989; Novakowski, 1989; and Hyder et al., 1994). In addition, the general relationship between pumping test and slug test models (Ramey and Agarwal, 1972; Peres et al., 1989) can be used to generate Stehfest algorithm slug test mathematical modeling equations with Stehfest algorithm pumping test mathematical modeling equations for partially penetrating wells in confined nonleaky, confined leaky, confined fissure and block, and unconfined aquifer conditions. Under favorable conditions when the magnitude of normalized heads during late time portions of the slug test are appreciable, it is possible to estimate the confining unit vertical hydraulic conductivity under confined leaky aquifer conditions and the vertical to horizontal hydraulic conductivity ratio under unconfined aquifer conditions.

### Confined Aquifer Laplace–Transform Equations

The slug test Laplace–transform solution for a slugged well in an infinite confined nonleaky aquifer with partially penetrating wells and wellbore storage is as follows (Dougherty, 1989, pp. 567–568):

$$\overline{h}_D/H_o = C_D\rho(A_o + A)/[1 + pC_D\rho(A_o + A)] \tag{3.253}$$

where

$$A_o = K_0(p^{1/2})/[p^{1/2}K_1(p^{1/2})] \tag{3.254}$$

$$A = 2/\rho^2 \sum_{n=1}^{\infty} \{[\sin(n\pi z_{bD}/b_D) - \sin(n\pi z_{aD}/b_D)]^2\ K_0[(p + n^2\ \pi^2/b_D{}^2)^{0.5}]\}/ \{n^2\pi^2\ (p + n^2\pi^2/b_D{}^2)^{0.5}\ K_1[(p + n^2\ \pi^2/b_D{}^2)^{0.5}]\} \tag{3.255}$$

$$C_D = r_c{}^2/(2\ r_w{}^2\ S\ \rho) \tag{3.256}$$

$$\rho = (z_b - z_a)/b \tag{3.257}$$

$$z_{aD} = z_a/r_w \tag{3.258}$$

$$z_{bD} = z_b/r_w \tag{3.259}$$

$$b_D = b/r_w \tag{3.260}$$

$r_w$ is the pumped well effective radius, $S$ is the aquifer storativity, $r_c$ is the pumped well casing radius, $p$ is the Laplace–domain variable, $H_o$ is the initial displacement from static head, $z_a$ is the distance from the aquifer base to the slugged well screen base, $z_b$ is the distance from the aquifer base to the slugged well screen top, and $b$ is the aquifer thickness.

The slug test Laplace–transform solution for an observation well in an infinite confined nonleaky aquifer with fully penetrating wells and no observation wellbore storage is as follows (Novakowski, 1989, p. 2379):

$$\overline{h}_D/H_o = K_0(r_D p^{1/2})/\{p^{1/2}[p^{1/2}K_0(p^{1/2}) + (1/C_D)K_1(p^{1/2})]\} \qquad (3.261)$$

where

$$r_D = r^2/r_w^2 \qquad (3.262)$$

$r$ is the distance between the slugged and observation wells and $r_w$ is the pumped well effective radius.

A numerical inversion program TYPCURV was developed by Novakowski (1990) to generate slug test observation well dimensionless time-normalized head values.

## Pumping-Slug Test Relationship

Pumping and slug test responses are related by the following dimensionless equation (Ramey and Agarwal, 1972; Peres et al., 1989):

$$H/H_0(t_D,r_D,C_D) = C_D[dp_D/dt_D(t_D,r_D,C_D)] \qquad (3.263)$$

where

$$t_D = Tt/(r^2S) \qquad (3.264)$$

$$r_D = r/r_w \qquad (3.265)$$

$$C_D = r_c^2/(2r_w^2S) \qquad (3.266)$$

$$p_D = 2\pi Ts/Q \qquad (3.267)$$

$T$ is the aquifer transmissivity, $t$ is the elapsed time, $S$ is the aquifer storativity, $p_D$ is the pumping test dimensionless drawdown, $H/H_0$ is the slug test normalized head, $r_w$ is the slugged well effective radius, $r_c$ is the slugged well casing radius, $r$ is the distance between the axis of the pumped or slugged well and the observation point ($r_w$ is substituted for $r$ in the case of the pumped or slugged well), $dp_D/dt_D(t_D,r_D,C_D)$ is the first derivative of the dimensionless pumping test drawdown with respect to the first derivative of the dimensionless time $t_D$, $s$ is the drawdown, and $Q$ is the pumped well constant discharge rate.

Thus, slug test time-normalized heads are first derivatives (slopes) of the dimensionless pumping test time drawdowns calculated with Stehfest algorithm pumping test mathematical modeling equations multiplied by $C_D$. First derivatives of pumping test dimensionless time drawdown are calculated with an algorithm listed by Bourdet et al. (1989). That algorithm calculates the first derivative of head change with respect to the natural logarithm of the change in time. The derivative is averaged over time periods before and after the point of interest. The slopes of the head change vs. the change in time are weighted and the head derivative for the point of interest is calculated with the following equation:

$$(dH/dt)_i = [(\Delta H_1/\Delta t_l)\Delta t_2 + (\Delta H_2/\Delta t_2)\Delta t_1]/(\Delta t_1 + \Delta t_2) \tag{3.268}$$

where subscript 1 refers to the points before the points of interest $i$, subscript 2 refers to the points after the points of interest $i$, $(dH/dt)_i$ are the slopes of the pumped or observation well dimensionless head changes vs. the changes in time at the points of interest, $\Delta H_1$ is the pumped or observation well dimensionless head change over the interval between the point of interest $i$ and the point before the point of interest, $\Delta H_2$ is the pumped or observation well dimensionless head change over the interval between the point of interest $i$ and the point after the point of interest, $\Delta t_1$ is the dimensionless natural logarithmic time change over the interval between the point of interest $i$ and the point before the point of interest, and $\Delta t_2$ is the dimensionless natural logarithmic time change over the interval between the point of interest $i$ and the point after the point of interest.

Two derivative algorithm methods are supported in the program DERIV developed by Spane and Wurster (1993): fixed endpoint and least-squares fit. The fixed-endpoint method is usually used for calculating derivatives of type curve values that are relatively free of noise. The least-squares fit method is usually used for calculating derivatives of noisy test data. In the fixed-endpoint method, the points immediately before and after the specified time $L$ spacing from the point of interest are used in calculating mean slopes. The calculated slopes from the fixed endpoints to the point of interest are then weighted by multiplying each by its time distance to the point of interest, divided by the sum of the time distances to the two endpoints.

In the least-squares method, all data from the points immediately before and after the specified $L$ spacing are used in calculating the slopes to the left and right of the point of interest. The calculated slopes are then weighted as described for the fixed-endpoint method. The $L$ spacing may range from 0 to 5 (Spane and Wurster, 1993). An $L$-spacing value of 0.2 is commonly used to reduce noise in the calculated derivative. Larger $L$-spacing values can lead to oversmoothing of data.

## Aquifer and Well Condition Effects

Aquifer type, well penetration, wellbore skin, and observation delayed response (wellbore storage) can appreciably affect slug test time-normalized heads. Slugged well time-normalized heads for different aquifer types (confined nonleaky, confined

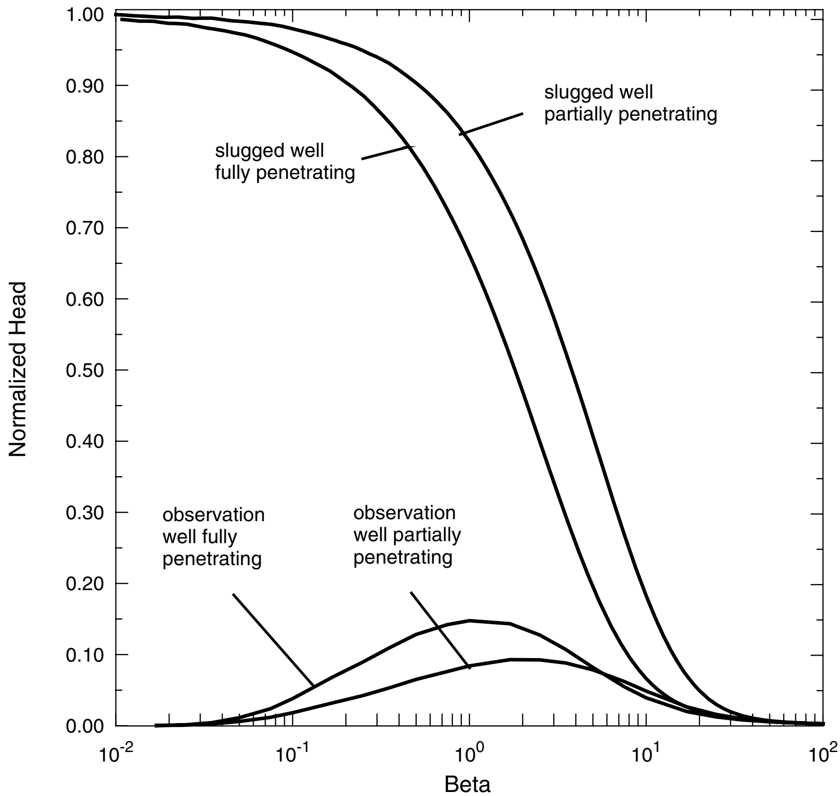



**FIGURE 3.5** Graph showing well partial penetration effects on slug test type curve values.

leaky, and confined fissure and block aquifers) with the same parameter values are nearly identical except for late times. Slugged well time-normalized heads for unconfined aquifers are appreciably offset from time-normalized heads for other aquifer types. Slugged well time-normalized heads for various ratios of aquifer vertical to horizontal hydraulic conductivity are similar in shape and closely spaced.

As demonstrated in Figure 3.5, time-normalized heads for a partially penetrating slugged well or a nearby observation well are shifted to the right of time-normalized heads for a fully penetrating slugged well. It is apparent that erroneously low hydraulic conductivity values are calculated by applying fully penetrating slugged well models to data for partially penetrating slugged wells.

Slugged well time-normalized heads can be significantly affected by wellbore skin. The hydraulic conductivity of the wellbore skin can either be larger or smaller than that of the formation. As demonstrated in Figure 3.6, time-normalized heads for a slugged well or a nearby observation well with wellbore skin whose hydraulic conductivity is less than that of the formation are shifted to the right of time-normalized heads for a slugged well with no wellbore skin. Erroneously

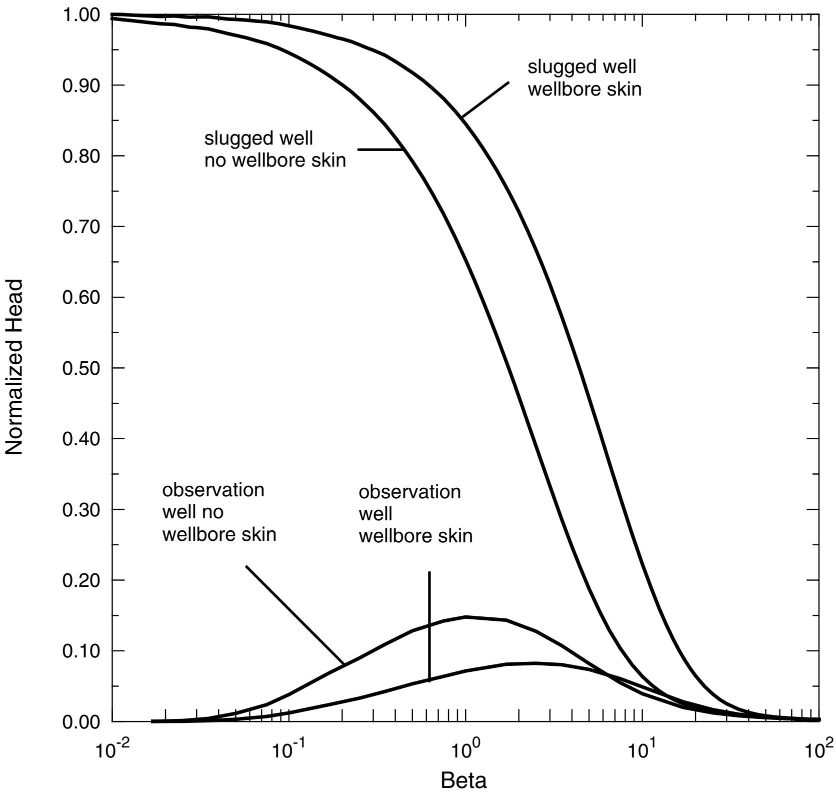



**FIGURE 3.6** Graph showing wellbore skin effects on slug test type curve values.

low hydraulic conductivity values are calculated by applying slugged well with no wellbore skin models to data for slugged wells with wellbore skin. An implausible low storativity estimate obtained with a Stehfest algorithm slug test model indicates the presence of wellbore skin.

Moench and Hsieh (1985, p. 20) present an equation for analysis of slugged well test data accounting for a skin of finite thickness. The equation assumes a confined nonleaky aquifer and a fully penetrating well. Families of type curves generated with that equation for different values of the ratio of the aquifer hydraulic conductivity to the skin hydraulic conductivity have nearly identical shapes except for very low ratios. Therefore, there will be a large degree of nonuniqueness in matching test data to a family of type curves. Accurate estimates of aquifer hydraulic conductivity cannot be obtained under most circumstances and it is not possible to tell whether there is a skin with a different hydraulic conductivity than that of the aquifer. Time-normalized head data for observation wells close to the slugged well (within tens of feet) are sufficiently different in shape and magnitude to allow a reasonable estimate of storativity or specific yield.

Slug test observation well type curve values with delayed response differ appreciably from type curve values with no delayed response as illustrated in

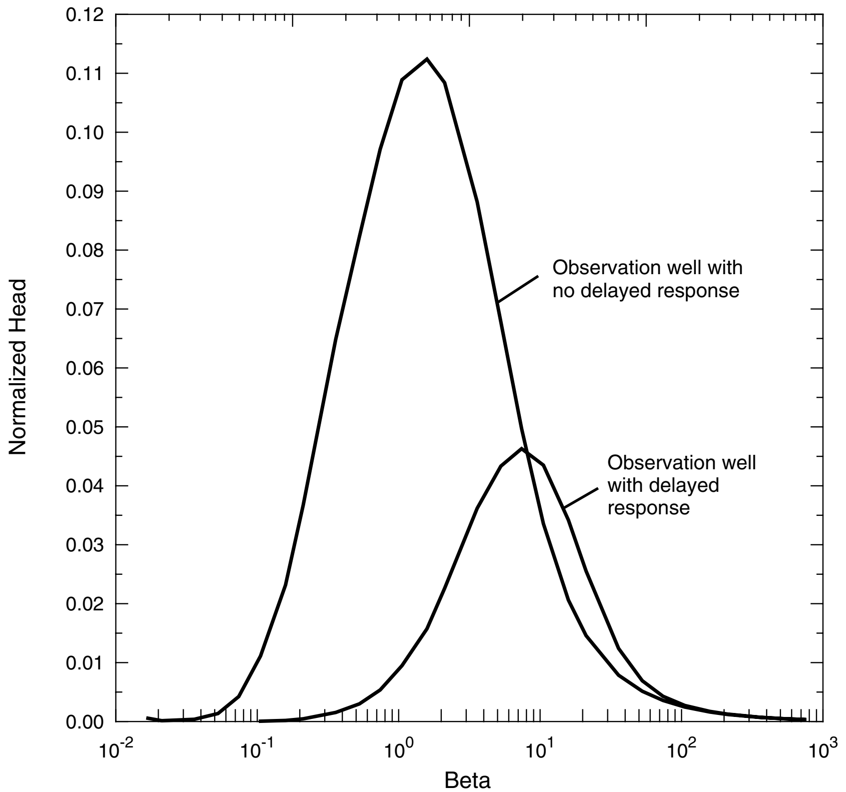



**FIGURE 3.7** Graph showing observation well delayed response effects on slug test type curve values.

Figure 3.7. Type curve values with delayed response have lower peaks and are shifted to the right of type curve values without delayed response. Usually observation wells are packed off to eliminate delayed response.

## HORIZONTAL ANISOTROPY

Some sedimentary and fractured aquifers are horizontally anisotropic. The horizontal hydraulic conductivity in one direction may be between 2 and 20 times or more the horizontal hydraulic conductivity in another direction. Drawdown contours around a pumped well in an anisotropic aquifer form concentric ellipses rather than circles, as they would in an isotropic aquifer. Major and minor directions of transmissivity coincide with major and minor ellipse axes. Simulation of horizontal anisotropy with numerical mathematical modeling equations can be accomplished by varying finite-difference grid cell hydraulic characteristics. Simulation is more difficult with analytical mathematical modeling equations as described below.

Aquifer test data for severe horizontal anisotropy conditions can be analyzed with the parallel boundary (strip aquifer) image well theory. Aquifer test data for less severe horizontal anisotropy can be analyzed with analytical methods derived by Hantush (1966a and 1966b) as explained in Kruseman and de Ridder (1991) and Batu (1998). These analytical methods cover the three following horizontal anisotropy conditions:

1. Principal directions of horizontal anisotropy known and ellipse of equal drawdown unknown
2. Principal directions of horizontal anisotropy unknown and ellipse of equal drawdown unknown
3. Ellipse of equal drawdown known

Time-drawdown data are matched to an appropriate family of type curves for isotropic conditions in the case of horizontal anisotropy Condition 1 or Condition 2. The effective transmissivity ($T_e$) is calculated with isotropic analytical mathematical modeling equations and the dimensionless drawdown and measured drawdown match point coordinates. The values of $T_e$ calculated for all observation wells should be approximately the same. The average value of $T_e$ is used in Equation 3.269. The effective transmissivity ($T_e$) is defined by the following equation (Hantush, 1966):

$$T_e = (T_x T_y)^{0.5} \tag{3.269}$$

where $T_x$ is the transmissivity in the major direction of horizontal anisotropy and $T_y$ is the transmissivity in the minor direction of horizontal anisotropy.

The ratio $S/T_n$, where $S$ is storativity and $T_n$ is the transmissivity in a direction that makes an angle ($\ominus + \alpha$) with the $X$ axis (major axis) as defined in Figure 3.8, is calculated with isotropic analytical mathematical modeling equations and the dimensionless time and measured time match point coordinates. The storativity, $T_x$, and $T_y$ can be calculated provided there are one or more observation wells or piezometers on more than one ray of observation wells or piezometers. If the principal directions of horizontal anisotropy are known, two observation wells or piezometers on different rays are sufficient. If the principal directions of horizontal anisotropy are unknown, three observation wells or piezometers on different rays are required.

$T_n$ is defined by the following equation (Hantush,1966):

$$T_n = T_x/[\cos^2(\ominus + \alpha_n) + m\sin^2(\ominus + \alpha_n)] \tag{3.270}$$

where $n$ is the ray number and

$$m = T_x/T_y = (T_e/T_y)^2 \tag{3.271}$$

$$\alpha_1 = 0 \tag{3.272}$$

$$T_1 = T_x/(\cos^2\ominus + m\sin^2\ominus) \tag{3.273}$$

For confined nonleaky and unconfined aquifers:

$$a_n = T_1/T_n = [\cos^2(\ominus + \alpha_n) + m\sin^2(\ominus + \alpha_n)]/(\cos^2\ominus + m\sin^2\ominus) \tag{3.274}$$

where

$$a_1 = 1 \tag{3.275}$$

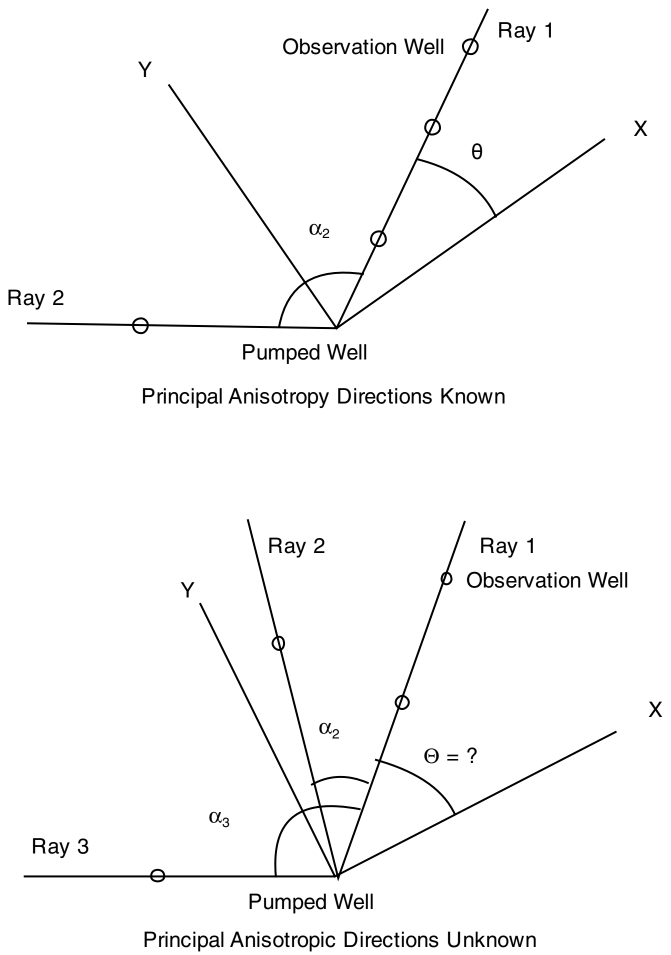



**FIGURE 3.8** Horizontal anisotropy parameters.

For confined leaky aquifers:

$$a_n = 0.5[T_1/T_n + (B'_1/B'_n)^2] \tag{3.276}$$

where

$$B'_n = (T_n b'/K')^{0.5} \tag{3.277}$$

$b'$ is the confining unit thickness and $K'$ is the confining unit vertical hydraulic conductivity from Equation 3.271 and Equation 3.274.

$$m = (T_e/T_y)^2 = [a_n\cos^2\ominus - \cos^2(\ominus + \alpha_n)]/[\sin^2(\ominus + \alpha_n) - a_n\sin^2\ominus] \tag{3.278}$$

If the principal directions of anisotropy are not known, $\ominus$ is calculated with the following equation:

$$\begin{aligned}\tan(2\ominus) = -2\{&[(a_3 - 1)\sin^2\alpha_2 - (a_2 - 1)\sin^2\alpha_3]/\\ &[(a_3 - 1)\sin 2\alpha_2 - (a_2 - 1)\sin 2\alpha_3]\}\end{aligned} \tag{3.279}$$

Equation 3.279 has two roots in the range 0 to $2\pi$, one being the major axis of transmissivity ($X$ axis) and the other being the minor axis of transmissivity ($Y$ axis). If one root is $\delta$, the other will be $\delta + \pi$. The value of $\ominus$ that makes $m > 1$ locates the major axis of anisotropy. A negative value of $\ominus$ indicates the positive $X$ axis lies to the left of the first ray of observation wells or piezometers.

If the principal directions of anisotropy are known, $a_2$ is calculated with Equation 3.274 or Equation 3.276 and previously calculated ratios $S/T_n$. Values of $\ominus$, $\alpha_2$, $a_2$, and $T_e$ are substituted into Equation 3.278 to calculate $m$. Values of $T_e$ and $m$ are substituted into Equation 3.271 to calculate $T_y$ and $T_x$. Values of $T_x$, $m$, $\ominus$, and $\alpha_2$ are substituted into Equation 3.273 and Equation 3.274 or Equation 3.276 to calculate $T_1$ and $T_2$. Finally, values of $S/T_1$, $S/T_2$, $T_1$, and $T_2$ are used to calculate $S$.

If the principal directions of anisotropy are unknown, $a_2$ and $a_3$ are calculated with Equation 3.274 or Equation 3.276 and previously calculated ratios $S/T_n$. Values of $a_2$, $a_3$, $\alpha_2$, and $\alpha_3$ are substituted into Equation 3.277 to calculate $\ominus$. Values of $\ominus$, $T_e$, $\alpha_2$, and $a_2$ (or $\alpha_3$ and $a_3$) are substituted into Equation 3.278 to calculate $m$. Values of $T_x$, $m$, and $\ominus$ and the values of $\alpha_1 = 0$, $\alpha_2$, and $\alpha_3$ are substituted into Equation 3.270 to calculate $T_1$, $T_2$, and $T_3$. Finally, values of $S/T_1$, $S/T_2$, $S/T_3$, $T_1$, $T_2$, and $T_3$ are used to calculate $S$.

A Fortran computer program, Tensor2D, developed and documented by Maslia and Randolph (1987) can be used to analyze pumping test data for an anisotropic confined nonleaky aquifer. Tensor2D is based on the equation of drawdown formulated by Papadopulos (1965) for nonsteady flow in an infinite anisotropic confined nonleaky aquifer. Data for more than three observation wells or piezometers can be analyzed with a weighted least-squares optimization procedure. Several other methods for analyzing pumping test data in anisotropic aquifers are described in the literature (Way and McKee, 1982; Neuman et al., 1984; and Hsieh et al., 1985).

## HORIZONTAL HETEROGENEITY

Many aquifers are horizontally heterogeneous. For example, aquifer hydraulic conductivity can progressively increase or decrease due to major depositional regimes. The aquifer stratigraphic framework can consist of several beds of different hydraulic conductivities. There can be sharp contrasts in aquifer hydraulic conductivity over limited distances. Aquifers can be trending, layered, and discontinuous. Aquifer heterogeneities can follow the complex spatial distribution of structural or sedimentologic architectural elements.

Pumping test data for horizontal heterogeneous aquifers are commonly analyzed analytically with one of the following two methods:

1. Interpret time-drawdown data for far wells to estimate effective (representative) large-scale hydraulic parameter values with homogeneous aquifer analytical equations
2. Interpret time-drawdown data for the pumped and near observation wells to estimate small-scale hydraulic parameter values with homogeneous aquifer analytical equations

Method 1 is primarily important for water supply studies. In Method 1, heterogeneities are simulated by averaging small-scale spatial hydraulic parameter variations. Method 2 is primarily important for contamination transport studies because contaminant spreading largely depends on spatial variations in hydraulic conductivity. Analytical pumping test analysis in heterogeneous aquifers is much more complex than analytical pumping test analysis in homogeneous aquifers because the sensitivity of drawdown and recovery data to heterogeneity and anisotropy varies both in space and time.

Oliver (1993) and Leven (2002) describe drawdown data sensitivity to heterogeneity during pumping tests briefly as follows:

- Drawdown is sensitive to heterogeneity within the pumping test domain, which expands in volume with time, the area of influence of heterogeneity on drawdown is elliptical in shape, and the influence is not spatially uniform within the pumping test domain.
- Drawdown is most sensitive to heterogeneity within the pumping test domain during early times.
- Drawdown sensitivity to heterogeneity depends on the contrast of hydraulic conductivity.
- Drawdown sensitivity to heterogeneity is highest close to the pumped well and the heterogeneity.

- Drawdown sensitivity to anisotropy depends on the relative locations of the principal axis of anisotropy and the pumped and observation wells.
- Hydraulic conductivity heterogeneity can either increase or decrease drawdown depending on the relative locations of the heterogeneity and the pumped and observation wells.
- Increased storativity heterogeneity decreases drawdown and decreased storativity heterogeneity increases drawdown regardless of the relative locations of the heterogeneity and pumped and observation wells.

Sanchez-Vila (1999) studied the results of analytically analyzing pumping test data for heterogeneous aquifers assuming horizontal homogeneous aquifer (Jacob's method) conditions. Briefly, Sanchez-Vila's conclusions are as follows:

- Variability in transmissivity is apparent as a variability in storativity.
- Hydraulic conductivity values calculated with late time-drawdown data for several fully penetrating observation wells tend to be uniform in space and represent the effective (geometric mean) hydraulic conductivity within the pumping test domain.
- Storativity values calculated with late time-drawdown data for several fully penetrating observation wells tend to be variable in space and are not representative by themselves.
- Real storativity can rarely be obtained by analyzing pumping test data for heterogeneous aquifers assuming horizontal homogeneous aquifer (Jacob's method) conditions.

It follows that variations in calculated storativity can be useful in diagnosing heterogeneity. Studies of data for observation wells at variable distances from the pumped well and locations on rays along and at right angles to heterogeneities also can be useful in diagnosing heterogeneity. For example, average hydraulic parameter values for the aquifer test domain can be calculated with late time-drawdown data. Theoretical distance-drawdown data can then be determined with these values and compared with measured distance-drawdown data. Deviations between theoretical and measured distance-drawdown data represent the effects of heterogeneity.

The effects of heterogeneity also can be diagnosed with pumped well specific capacity data for a selected time. As a result of heterogeneity, measured specific capacity in the pumped well, assuming complete well development and negligible well losses, will differ from theoretical specific capacity based on the average hydraulic parameter values calculated with aquifer test late time-drawdown data.

Heterogeneity also can be diagnosed with composite plots of time-drawdown data for pumped and observation wells. Evaluation of time-drawdown data observed at different locations in an aquifer may not result in one consistent set of hydraulic parameter values, which indicates that the aquifer is not homogeneous.

Several analytical time-drawdown approaches to aquifer test analysis in heterogeneous aquifers have been developed in recent decades (see Butler, 1988;

Butler, 1990; Butler, 1991; Butler and Liu, 1991; Butler and Liu, 1993; Indelman et al., 1996; Kabala, 2001; Leven, 2002; Oliver, 1993; Sanchez-Vila, 1997; Sanchez-Vila et al., 1999; Vasco et al., 2000; Yeh, 1986; Zlotnik and Ledder, 1996; and Walker and Roberts, 2003). Some of the approaches such as that of Butler (1988) are based on specified patterns of heterogeneity. It is usually assumed that transmissivity varies logarithmically in space while storativity is constant in space. Other approaches such as that of Oliver (1993) do not predefine the pattern of heterogeneity. The practical application of these methods has been limited.

Oliver (1993) presents an unsteady state flow analytical solution for transmissivity and storativity with constant pumping from a single well in a radially symmetric modest heterogeneous aquifer with transmissivity varying logarithmically in space and uniform storativity. The solution involves Frechet kernels (sensitivity coefficients) as convolution integrals in the time domain, which must be evaluated numerically. Knight and Kluitenberg (2005) present explicit analytical expressions for storativity and transmissivity Frechet kernels for both pumping and slug tests in a radially symmetric modest heterogeneous aquifer with transmissivity and storativity varying uniformly in space. The explicit analytical expressions involve Bessel functions.

Sanchez-Vila (1997) presents a steady state flow analytical solution for the effective transmissivity with constant pumping from a single well of finite radius in a heterogeneous statistically isotropic random aquifer. The solution indicates effective transmissivity is an increasing monotonic function of distance from the pumped well. Effective transmissivity rises from the harmonic mean of the point values close to the pumped well and tends asymptotically toward the geometric mean far from the pumped well.

Leven (2002) describes a consecutive multiple well approach to pumping test analysis in heterogeneous aquifers wherein several wells are located within and along the heterogeneity. The wells have small diameters and are completely developed. A constant low rate pumping test is conducted consecutively at each well. The duration of each test is short and usually 1000 sec or less. Time-drawdown data for each pumped well are plotted as semilogarithmic graphs.

Straight lines are drawn through the very early time-drawdown data when wellbore storage effects are negligible. The straight lines are extended to zero drawdown. The slopes of the straight lines and zero drawdown intercepts are used to calculate small scale values of hydraulic conductivity and storativity. These values are assigned to a pumping test domain within a radius of tens of feet of each pumped well. The time when wellbore storage effects are negligible is ascertained by noting that pumped well time-drawdown data plot as a straight line with a slope of one on a double-logarithmic time-drawdown graph during the period when wellbore storage effects are appreciable.

The slope of late time-drawdown data differs from the slope of the early time-drawdown data in heterogeneous aquifers. Large-scale effective values of hydraulic conductivity and storativity are calculated with late time-drawdown data and assigned to a pumping test domain within a radius of hundreds to thousands of feet of the pumped wells. The quality of hydraulic conductivity and storage values depends on the effectiveness of well development.

There are other approaches for analyzing pumping test data for heterogeneous aquifers. For example, information concerning aquifer heterogeneity can be obtained with large drawdown slug tests (Gonzalo Pulido, HydroQual, Inc. gpulido@hydroqual.com). Large drawdown slug tests are slug tests with large normalized heads greater than 5 m, which enhance the estimation of hydraulic parameters with data from observation wells tens of meters from the slugged well.

Vertical variations in hydraulic conductivity can be estimated with dipole flow tests as described by Butler (1998a). The dipole flow test involves a single well test in which a three-packer tool is placed in the screened (open) interval of a well. A small downhole pump moves water from one chamber of the tool to the other through the center of the middle packer, thereby setting up a circulation pattern in the adjacent aquifer. The head difference between the two chambers is used to estimate the hydraulic conductivity of near-well portions of the aquifer.

Spatial variations in hydraulic conductivity can be estimated with the direct-push method (Butler et al., 2000). This method involves performing series of slug tests in direct-push rods as the rods are driven progressively deeper into the formation. Screened intervals in the rods are only exposed to the aquifer during slug tests, thereby minimizing the amount of well development without the need of permanent wells.

Interwell variations in horizontal hydraulic conductivity can be estimated with the hydraulic tomography approach (Bohling et al., 2003; Bohling et al., 2002; Butler et al., 1999; and Yeh and Liu, 2000). This approach consists of a series of short-term pumping tests stressing different vertical aquifer intervals in networks of multilevel small-diameter sampling wells. Detailed drawdown data is obtained with miniature fiber-optic pressure sensors or air-pressure transducers. Data from all tests are analyzed simultaneously to characterize the hydraulic conductivity variation between wells.

## IMAGE WELL THEORY

According to the image well theory (see Ferris et al., 1962, pp. 144–146 and Walton, 1962), a full barrier boundary is defined as a line (streamline) across which there is no flow, and it may consist of folds, faults, or relatively impervious deposits such as shale or clay. A full recharge boundary is defined as a line (equipotential) along which there is no drawdown, and it may consist of increased aquifer transmissivity or streams, lakes, and other surface water bodies hydraulically connected to the aquifer. Most full hydrogeologic boundaries are not clear-cut straight-line features but are irregular in shape and extent. However, complicated full boundaries are simulated with straight-line demarcations.

The image well theory for a full barrier boundary can be stated as follows: The effect of a full barrier boundary on the drawdown in a well, as a result of pumping from another well, is the same as though the aquifer were infinite in areal extent and a like discharging image well were located across the full barrier boundary on a perpendicular line thereto and at the same distance from the full

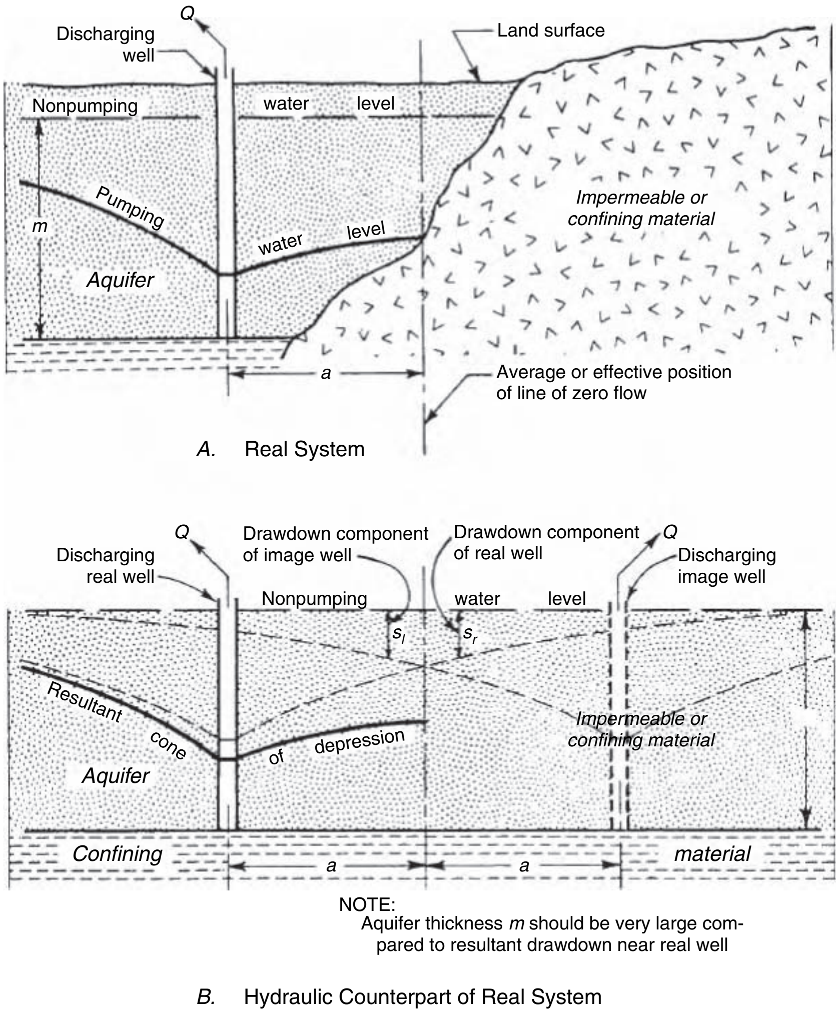



**FIGURE 3.9** Image well system for barrier boundary (from Ferris et al., 1962, U.S. Geological Survey, Water-Supply Paper 1536E).

barrier boundary as the pumped well as shown in Figure 3.9. The principle is the same for a full recharge boundary except the image well is assumed to be recharging the aquifer system instead of pumping from it as shown in Figure 3.10.

The image well, like the pumped well, can have wellbore storage and can partially penetrate the aquifer. The observation well wellbore storage is influenced by the image well. Thus, the impacts of full hydrogeologic boundaries on drawdown can be simulated by use of hypothetical wells. Full boundaries are replaced by imaginary wells that produce the same disturbing effects as the boundaries. Full boundary well hydraulics problems are thereby simplified to consideration of an infinite aquifer system in which real and image wells operate simultaneously.

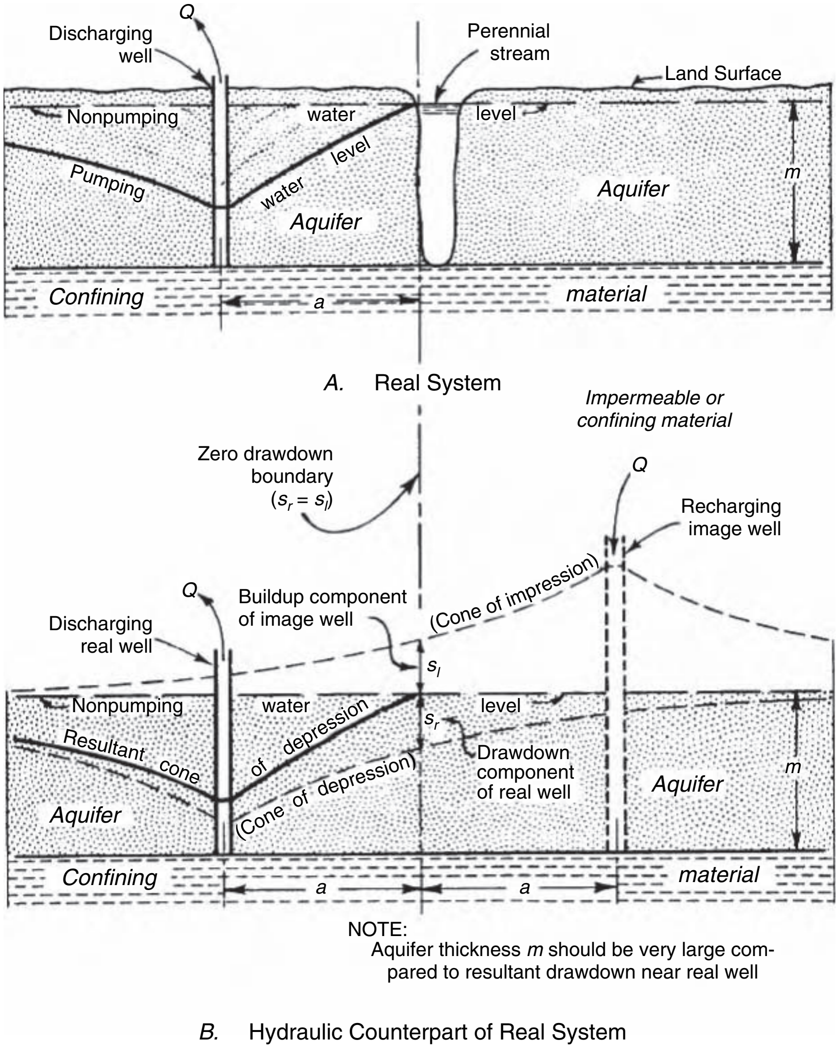



**FIGURE 3.10** Image well system for recharge boundary (from Ferris et al., 1962, U.S. Geological Survey, Water-Supply Paper 1536E).

Total drawdown at any particular time is the algebraic summation of real and image well drawdown or buildup components.

A partial barrier boundary (barrier discontinuity) is defined as a line beyond which the aquifer transmissivity is much less than the aquifer transmissivity at the pumped well. A partial recharge boundary (recharge discontinuity) is defined as a line beyond which the aquifer transmissivity is much greater than the aquifer transmissivity at the pumped well. Most discontinuities are not clear-cut straight-line features but are irregular in shape and extent. However, complicated discontinuities are simulated with straight-line demarcations.

The image well theory for a discontinuity can be stated as follows: The effect of a barrier discontinuity on the drawdown in a well, as a result of pumping from another well, is the same as though the aquifer were infinite in areal extent and a discharging image well were located across the barrier discontinuity on a perpendicular line thereto and at the same distance as the pumped well. The image well discharge is a fraction of the pumped well discharge and depends on the relative aquifer transmissivities on both sides of the boundary (Muskat, 1937; Streltsova, 1988, p. 219; and McKinley and Streltsova, 1993, p. 130).

The principle is the same for a recharge discontinuity except the image well is assumed to be recharging the aquifer system instead of pumping from it. The image well, like the pumped well, has wellbore storage and can partially penetrate the aquifer. The observation well can have wellbore storage in response to the influence of the image well. Thus, the impacts of discontinuities on drawdown can be simulated by use of hypothetical wells. Discontinuities are replaced by imaginary wells that produce the same disturbing effects as the discontinuities. Discontinuity well hydraulics problems are thereby simplified to consideration of an infinite aquifer system in which real and image wells operate simultaneously. Total drawdown at any particular time is the algebraic summation of real and image well drawdown or buildup components.

The image well strength with a discontinuity can be estimated with the following equation (Muskat, 1937):

$$Q_{is} = QD_{is} \tag{3.280}$$

where

$$D_{is} = (T_p - T_d)/(T_p + T_d) \tag{3.281}$$

$Q_{is}$ is the constant image well strength, $Q$ is the constant pumped well discharge rate, $T_p$ is the aquifer transmissivity between the pumped well and the discontinuity, and $T_d$ is the aquifer transmissivity beyond the discontinuity.

Strictly speaking, equations 3.280 and 3.281 are valid only when the diffusivities ($T/S$) on either side of the discontinuity are equal. However, for practical purposes, the use of these equations results in reasonable discontinuity simulations. The exact equation when the diffusivities are unequal is presented by Streltsova (1988, pp. 219–220). McKinley and Streltsova (1993, p. 130) provide a monograph for analyzing a discontinuity when the diffusivities on either side of the discontinuity are unequal.

Nind (1965) presents equations for drawdown in the presence of linear discontinuities. Nonsteady state equations describing drawdown on both sides of a discontinuity in a confined nonleaky aquifer were derived by Fenske (1984).

With barrier boundaries, water levels in observation wells decline at an initial rate under the influence of the pumping well only, as if the aquifer system were infinite in areal extent. When the cone of depression of the boundary image well appreciably impacts the observation wells, the time rate of drawdown increases because the total rate of withdrawal from the aquifer system is then equal to that of the pumping well plus that of the discharging image well. Thus, the time-drawdown curve is deflected downward.

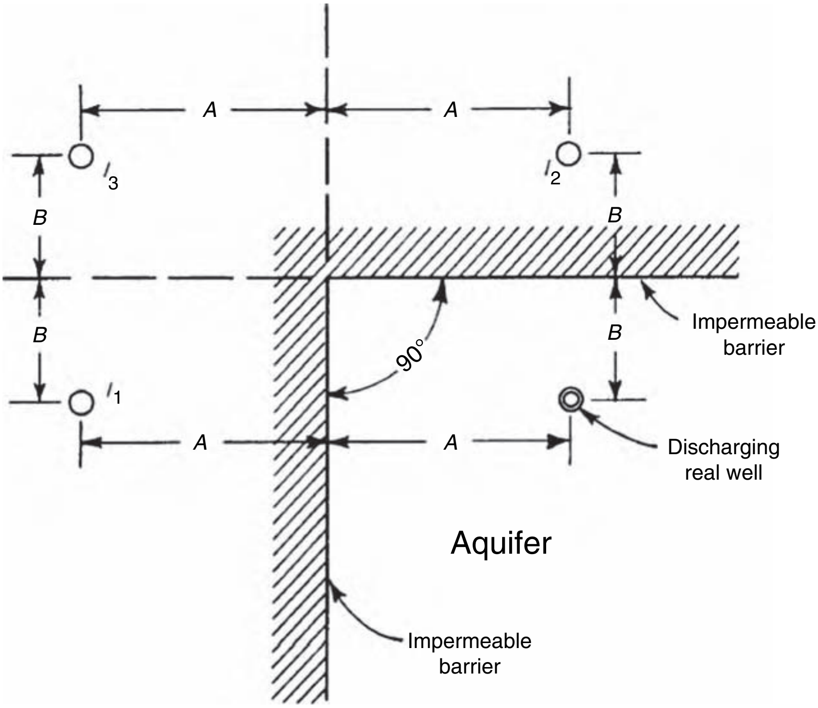
With recharge boundary conditions, water levels in observation wells decline at an initial rate under the influence of the pumping well only, as if the aquifer were infinite in areal extent. When the cone of impression of the recharging image well appreciably impacts the observation well, the time rate of drawdown changes and decreases. With a full recharge boundary, equilibrium conditions will eventually prevail and the time-drawdown curve will level off.

When a well near a stream hydraulically connected to an aquifer is pumped, the cone of depression grows until it intercepts sufficient area of the streambed and is deep enough beneath the streambed so that induced streambed infiltration balances discharge. The cone of depression may expand only partially or across and beyond the streambed depending upon the hydraulic conductivity of the streambed. The use of the image well theory to simulate induced streambed infiltration assumes that streambed partial penetration and aquifer stratification are integrated into the effective distance to the recharging image well.

Aquifers are often delimited by two or more boundaries as shown in Figure 3.11. Two converging boundaries delimit a wedge-shaped aquifer, two parallel boundaries delimit an infinite-strip aquifer, two parallel boundaries intersected at right angles by a third boundary delimit a semi-infinite strip aquifer, and four boundaries intersecting at right angles delimit a rectangular aquifer. The image well theory is applied to such cases by taking into consideration successive image well reflections on the boundaries.

A number of image wells are associated with a pair of converging boundaries. A primary image well placed across each boundary balances the impacts of the pumping well at each boundary. However, each primary image well produces an unbalanced impact at the opposite boundary. Secondary image wells must be added at appropriate positions until the impacts of the pumping and primary image wells are balanced at both boundaries. Although image well systems can be devised regardless of the wedge angle involved, simple solutions of closed image well systems are preferred. The actual aquifer wedge angle is approximated as equal to one of certain aliquot parts of 360 degrees. These approximate angles were specified by Ferris et al. (1962, p.154) as follows: If the aquifer wedge boundaries are of like character, the approximate angle must be an aliquot part of 180 degrees; if the aquifer wedge boundaries are not of like character, the approximate angle must be an aliquot part of 90 degrees; and if the pumping well is on the bisector of the wedge angle and the aquifer wedge boundaries are like in character and both barriers, the approximate angle must be an odd aliquot part of 360 degrees. Under these conditions, the exact number of image wells is equal to (360 degrees divided by the wedge angle) minus 1.

The character of each image well is the same if the aquifer wedge boundaries are of like character. If the aquifer wedge boundaries are not of like character, the character of each image well is ascertained by balancing the image well system considering each boundary separately with the following rules (Walton,




NOTES:
Image wells, *I*, are numbered in the sequence in which they were considered and located

Open circles signify discharging wells

**FIGURE 3.11** Image well system for wedge boundary (from Ferris et al., 1962, U.S. Geological Survey, Water-Supply Paper 1536E).

1963, pp. 20–21): A primary image well placed across a barrier boundary is discharging in character, and a primary image well placed across a recharge boundary is recharging in character; a secondary image well placed across a barrier boundary has the same character as its parent image well, and a secondary image well placed across a recharge boundary has the character opposite that of its parent image well.

Two parallel boundaries require the use of an image well system extending to infinity. Each successively added secondary image well produces a residual impact at the opposite boundary. However, in practice it is only necessary to add pairs of image wells until the next pair has negligible influence (< 0.01 ft) on the sum of all image well impacts out to that point.

If $s_t$ is the total drawdown in an observation well at time $t$, $s_p$ is the component of drawdown caused by the pumped well at time $t$, and $s_I$ is the component of drawdown or buildup caused by an image well associated with a single boundary at time $t$, then

$$s_t = s_p + s_I \text{ for a barrier boundary} \tag{3.282}$$

$$s_t = s_p - s_I \text{ for a recharge boundary} \tag{3.283}$$

Appropriate dimensionless drawdowns are utilized to calculate $s_p$ and $s_I$ depending on existing aquifer conditions. For example, with confined nonleaky aquifer conditions:

$$s_o = QW(u)/(4\pi T) + Q_I W(u_I)/(4\pi T) \text{ for a barrier boundary} \tag{3.284}$$

$$s_o = QW(u)/(4\pi T) - Q_I W(u_I)/(4\pi T) \text{ for a recharge boundary} \tag{3.285}$$

where

$$u = r^2 S/(4Tt) \tag{3.286}$$

$$u = r_i^2/(4Tt) \tag{3.287}$$

$Q$ is the pumped well discharge rate, $T$ is the aquifer transmissivity, $r$ is the distance from the pumped well to the observation well, $S$ is the aquifer storage, $t$ is the elapsed time, $r_i$ is the distance from the image well to the observation well, $W(u)$ is the dimensionless drawdown for the pumped well and $W(u_I)$ is the dimensionless drawdown or recovery for the boundary image well.

## NUMERICAL MATHEMATICAL MODELING EQUATIONS

Numerical groundwater flow mathematical modeling equations can generate dimensionless or dimensional time-drawdown values for simplistic conceptual models equally as well as analytical models. In addition, numerical models can generate dimensionless and dimensional time-drawdown values for complex conceptual models. Both numerical and analytical models can simulate a homogeneous medium with a uniform thickness, confining unit storativity, delayed gravity drainage under unconfined aquifer conditions, wellbore storage, partially penetrating wells, boundaries as straight-line demarcations, single aquifer and confining unit, and a pumped well open to only one aquifer. Numerical models can simulate a heterogeneous aquifer with a nonuniform thickness, irregular boundaries, multiple aquifer and confining unit layers, and a pumped well open to several aquifers.

Numerical models are based on the following partial-differential equation describing the three-dimensional movement of groundwater of constant density through heterogeneous and anisotropic porous earth material under nonequilibrium conditions (McDonald and Harbaugh, 1988, p. 2-1):

$$\partial/\partial_x(K_{xx}\partial h/\partial x) + \partial/\partial_Y(K_{YY}\partial h/\partial y) + \partial/\partial_z(K_{zz}\partial h/\partial z) - W = S_s\partial h/\partial t \tag{3.288}$$

where $K_{xx}$, $K_{YY}$, and $K_{zz}$ are values of hydraulic conductivity along the $x$-, $y$-, and $z$-coordinate axes, which are assumed to be parallel to the major axes of hydraulic conductivity, $h$ is the potentiometric head, $W$ is a volumetric flux per unit volume and represents sources or sinks of water, $S_s$ is the specific storage of the porous material, and $t$ is time. The principal axes of hydraulic conductivity are assumed to be aligned with the coordinate directions. $S_s$, $K_{xx}$, $K_{YY}$, and $K_{zz}$ may be functions of space [$S_s = S_s(x, y, z)$, $K_{xx} = K_{xx}(x, y, z)$, etc.] and $W$ may be a function of space and time [$W = W(x, y, z, t)$].

The mathematical modeling equations describing groundwater flow consist of the partial-differential equation together with specification of flow or head conditions at the boundaries of an aquifer system and specification of initial head conditions. Numerical mathematical modeling equations commonly utilize the finite-difference approximation method (McDonald and Harbaugh, 1988; Anderson and Woessner, 1992).

## FINITE-DIFFERENCE APPROXIMATION METHOD

In the finite-difference approximation method, the continuous system described by partial-differential Equation 3.288 is replaced by a finite set of discrete points in space and time. Partial derivatives are replaced by terms calculated from the differences in head values at these points. Systems of simultaneous linear algebraic difference equations are generated and expressed as matrix equations and iterative numerical methods are used to solve the matrix equations. The solution of matrix equations leads to values of head at specific points and times.

The aquifer test conceptual model is replaced by a discretized grid of nodes centered at the pumped well and associated finite-difference cells (blocks) simulating one or more aquifer layers. Delayed gravity drainage under unconfined aquifer conditions is simulated with 10 or more confined layers and 1 unconfined layer. Parameter values are assigned to grid cell groups and boundary conditions are simulated along or within grid cell borders. Initial conditions are simulated by assigning the same head to all grid nodes. Aquifer test time and the pumping rate are discretized into small blocks of variable lengths to simulate wellbore storage.

The computer program MODFLOW, developed by the U.S. Geological Survey, is a prime example of the implementation of the finite-difference approximation method. MODFLOW requires several input files and produces several output files. Detailed instructions for the preparation of input files are provided in MODFLOW documentations.

## SOFTWARE SELECTION

A large variety of public domain and commercial aquifer test analysis software is available with a broad range of sophistication. Primary analytical and numerical software usually contains code to read input data files, code to calculate either or both dimensionless and dimensional time-drawdown values based on file input data (calculation engine), and code to generate output files for use with external paper graphs or external word processor, spreadsheet, database, graphics software, and automatic parameter estimation software. Primary software usually is distributed by governmental agencies or universities. Sophisticated analytical and numerical software contains code for interactive computer screen input (preprocessor), a calculation engine, internal automatic parameter estimation code, and code to display calculation results on the computer screen or with a printer (postprocessor). Sophisticated software can also contain integrated word processor, spreadsheet, database, and graphics capabilities for seamless analysis. Sophisticated software is usually distributed commercially. Less sophisticated software is included with some aquifer test analysis books.

The U.S. Geological Survey distributes the fully documented primary analytical software WTAQ described by Barlow and Moench (1999). WTAQ is written in Fortran and contains state-of-the-art code for calculating analytical Stehfest aquifer test model dimensionless or dimensional time-drawdown values with confined nonleaky or unconfined (water table) aquifer conditions.

The U.S. Geological Survey also distributes several versions of the fully documented primary numerical software MODFLOW described by McDonald and Harbaugh (1988). MODFLOW has become an international standard.

The WTAQ and MODFLOW software and documentation are available free of charge at water.usgs.gov/nrp/gwsoftware/.

Most analytical integral and numerical model commercial software can be purchased from the Scientific Software Group. P.O. Box 708188, Sandy, Utah 84070, (801) 208–3011 or at www.scisoftware.com.

Detailed information concerning commercial and free software can be obtained at:

www.groundwatermodels.com
www.flowpath.com
www.AQTESOLV.com
www.Aquifer-Test.com
typhoon.mines.edu/software/igwmcsoft/
www.rockware.com

The universal automatic parameter estimation software PEST and documentation and PEST Groundwater Data Utilities can be obtained free of charge at www.sspa.com/pest/.

The universal automatic parameter estimation software UCODE and documentation can be obtained free of charge at www.typhoon.mines.edu/software/igwmcsoft/.

The following books contain analytical integral aquifer test model software:

Dawson, K.J. and J.D. Istok. 1991. *Aquifer Testing: Design and Analysis of Pumping and Slug Tests*. Lewis Publishers, Boca Raton, FL at www.crcpress.com.

Halford, K.J. and E.L. Kuniansky. 2002. Spreadsheets for the Analysis of Aquifer-Test and Slug-Test Data. U.S. Geological Survey Open-File Report 02-197 at water.usgs.gov/pubs/of/.

Hall, Phil. 1996. *Water Well and Aquifer Test Analysis*. Water Resources Publications, LLC. Highlands Ranch, CO at www.wrpllc.com.

Boonstra, J. and R.A.L. Kselik. 2002. SATEM: Software for Aquifer Test Evaluation. Publication 57. International Institute for Land Reclamation and Improvement. The Netherlands at www.alterra-research.nl/pls.

Batu, Vedat. 1998. *Aquifer Hydraulics: A Comprehensive Guide to Hydrogeologic Data Analysis*. John Wiley Interscience Publications. Somerset, NJ.

Analytical Stehfest aquifer test model software is contained in the following book: Walton, W.C. 1996. *Aquifer Test Analysis with Windows Software*. Lewis Publishers, Boca Raton, FL at www.crcpress.com.

Both Fortran and Mathematica macros are provided in the following book: Cheng, A.H.-D. 2000. *Multilayered Aquifer Systems — Fundamentals and Applications*. Marcel Dekker, Inc., New York at www.amazon.com.

The following book gives instructions for ordering slug test analytical Stehfest aquifer test model software: Butler, J.J. 1998. *The Design, Performance, and Analysis of Slug Tests*. Lewis Publishers. Boca Raton, FL at www.crcpress.com.

The following book contains MODFLOW pre- and postprocessor software (PMWIN): Chiang, Wen-Hsing and Wolfgang Kinzelbach. 2003. 3D-*Groundwater Modeling with PMWIN*. Springer-Verlag, New York at www.uovs.ac.za/faculties.

The following book gives instructions for obtaining numerical pumping test software: Lebbe, L.C. 1999. *Hydraulic Parameter Identification — Generalized Interpretation Method for Single and Multiple Pumping Tests.* Springer-Verlag, New York at allserv.ugent.be/~luclebbe.